Virtual Underlying Security*

Robert Kast[†], André Lapied[‡], Sophie Pardo[§]

Abstract

Real option theory, that can be used for valuing public investments or solve optimal time schedule problems, is based on the existence of a relevant underlying security. In many applied works, there is no asset connected with obviousness to the risk to value and the difficulty is to determine such a relevant underlying asset. In this paper we propose a method for constructing a virtual underlying security as a portfolio of marketed assets, based on the functional correlation coefficient. Then, we tackle several problems arising in this replication procedure in an empirical setting.

Key words : Real options, risks, functional correlation coefficient.

JEL classification n°: C13, C14, D81, G12.

*We thank Professor Hans Föllmer for helpful discussion.

†CNRS-GREQAM-IDEP, kast@ehess.cnrs-mrs.fr

‡GREQAM-IDEP Université d'Aix-Marseille, groupe ESCMP, lapied@ehess.cnrs-mrs.fr

§GREQAM, pardo@ehess.cnrs-mrs.fr. This study was partly supported by the Conseil Régional PACA

Introduction

New theoretical approach to capital investments decisions of firms has been developed in the last decades answering to the traditional "Net Present Value "limitations. This called Real Options Theory allows to value some of real assets under uncertainty and is based on the analogy between a financial call option and investment opportunity (Myers [18], Kester [16]). It enables to take into account dynamic of investment decisions, and, particularly, irreversibilities and possibility to postpone choices. In this regard, it is possible to solve optimal time schedule problems and projects valuation (Dixit, Pendyck [7], Trigeorgis [22]).

Two techniques have emerged to formalize and solve these problems : dynamic programming and contingent claims valuation. The first is a general optimization method of an objective function that reflects decision maker's subjective valuation of risk when the second refers to the derivative assets pricing by arbitrage. These two approaches lead to very similar stochastic differential equations but do not require same assumptions. We are interested in the latter. The main requirement of contingent claim analysis is that stochastic changes in the assets we are trying to value must be spanned by existing assets in economy. That means the uncertainty over the future values of risk can be replicated by existing assets. This assumption of spanning should hold, by instance, for most commodities, which are traded on both spot and futures market. Anyway real-options theory was applied to copper mines, oil sinking, etc., for which exists a relation between a marketed asset and risk to value (*future* on copper, on oil[1]). Nevertheless, in many application ranges there is no asset connected, with obviousness, to one of the risk to value. For instance, Cortazar, Schwartz and Salinas[3] are concerned with the optimal environmental investment decision by a copper production firm that is confronted with governmental regulations. Two stochastic state variables intervene in the maximization program of the mine economic value : copper, for which exist both spot and future markets, and concentrate copper for which there is no market of future contracts. The authors assume that concentrate copper future prices risk can be spanned by existing securities in the economy. A portfolio of traded assets is referred to in order to replicate the future concentrates copper price process. That means that the portfolio returns are perfectly correlated with those of the concentrate. This portfolio is assumed to be a perfect hedge for the concentrate prices in absence of a future on concentrate and can be used in a real option approach to value the producing firm and/or determine an optimal time schedule. In other words, a deterministic function relating an underlying asset and an uncertain payoff process is sufficient to apply derivative asset pricing theory in the spirit of Black and Scholes [1] and Merton[17]. In applied works, the difficulty is to determine such an underlying asset. In this paper, we tackle several problems arising in this replication procedure in an empirical setting. We propose an optimization program based on the functional correlation

[1]See for instance Brennan, Schwartz[2], Slade[21]

coefficient (FCC), (see for instance Saporta [19]). Given past series of an asset payoffs to be replicated, and past series of traded asset payoffs we construct a portfolio by optimizing the empirical FCC. This yields at the same time an estimated deterministic function between the optimal portfolio and the asset payoffs.

Obviously, several problems arise, notably: Portfolio composition stability over time. The portfolio's composition must be checked to be stable over past time periods, with length equal to the future horizon. Stability is the condition for the portfolio to forecast the future payoffs of the risk over the given time horizon.

Non-unicity of replicating portfolio. The optimal portfolio will depend on a given set of traded assets. Modifying this set will modify the optimal portfolio although they follow perfectly correlated stochastic process. The asset market must then be checked to satisfy a non arbitrage condition.

In section one, we present the model and the procedure steps. Second part concentrates on the functional correlation coefficient and on the optimization method. Finally, in third section we apply the method on financial data and deal with the stability problem.

1 The model

Let $(\Omega, \mathcal{F}, P)$, be a filtered probability space. Let $Z(t), \tilde{Z}(t),\ 0 \leq t \leq T$, be two Brownian motions on $(\Omega, \mathcal{F}, P)$. Let X be the payoff process of a non-marketed asset to be valued and let Y be the payoff process of a marketed asset.

1.1 Uncertainty identification

In order to satisfy contingent claim analysis assumptions, the stochastic component of X returns must be replicated by existing assets (same probability law and perfectly correlated).

If we assume the existence of a deterministic function between X and Y at each date, then X can be considered as a derivative on the underlying Y. Finding such a Y and a deterministic function is a means to identify the source of uncertainty for X as a marketed one, from there on real options theory can be applied to X. We consider X and Y price processes, which dynamics are given by:

$$dX_t = \mu_X X_t dt + \sigma_X X_t d\tilde{Z}$$

$$dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dZ$$

and $Y(0) = y_0$[2]. dZ and $d\tilde{Z}$ are Gauss-Wiener processes increments such as $\mathcal{E}(dZ d\tilde{Z}) = \rho dt$ (ρ is constant correlation coefficient). We assume $\sigma_X > 0, \sigma_Y > 0$. Our problem is to identify

[2] Concerning X process, we will have initial or final conditions, according to the problem to be solved.

the Brownian motion generating X process. Alternatively, we look for Y such $\forall t$, there exists a deterministic function ϕ such as:

$$ln\left(\frac{X_{t+\Delta}}{X_t}\right) = \phi\left(ln\left(\frac{Y_{t+\Delta}}{Y_t}\right)\right)$$

The latter theorem characterizes a deterministic functional relation between prices (here, between log-prices) using the functional correlation coefficient.

Theorem 1.1
Let $\frac{dX}{X} = \mu_X dt + \sigma_X d\tilde{Z}$ *and let* $\frac{dY}{Y} = \mu_Y dt + \sigma_Y dZ$, *with constant correlation* ρ. *If* $\exists t$, $\exists \Delta$ *and* $\exists f$ *non-degenerated such as:*

$$\frac{X_{t+\Delta}}{X_t} = f\left(\frac{Y_{t+\Delta}}{Y_t}\right)$$

then $|\rho| = 1$

Proof: Proof is given in appendix 1. ■
Identifying the source of uncertainty for X amounts to say that the two processes are perfectly correlated.

1.2 Replication Portfolio

In case where there is no marketed asset Y connected with obviousness to X, we will construct Y as a portfolio of marketed assets, such that there exists a deterministic function ϕ. We consider prices of N financial traded assets $\{X^1, ..., X^N\}$ evolving according geometric Brownian processes:

$$dX_t^i = \mu_i X_t^i dt + \sigma_i X_t^i dZ_i$$

where μ_i are instantaneous trends, σ_i are standard deviations and dZ_i are increments to Gauss-Wiener processes such as $\mathcal{E}(dZ_i dZ_j) = \rho_{ij} dt$. With these assumptions the prices X_t^i are lognormally distributed and $\ln X_t^i$, has a conditional normal distribution with conditional mean and variance :

$$E_0(\ln X_t^i) = \ln X_0^i + (\mu_i - \tfrac{1}{2}\sigma_i^2)t$$
$$V_0(\ln X_t^i) = \sigma_i^2 t$$

We note $R_{it} = ln\left(\frac{X_t^i}{X_{t-1}^i}\right)$ the continuously compounded return of asset i at time t, that follows a conditional normal distribution with conditional mean and variance proportional to the length of the period.

To answer our replication problem, we construct:

$$R_{Yt} = \sum_{i=1}^{N} \lambda_i R_{it}$$

with $\sum_{i=1}^{N} \lambda_i = 1$. such that $\exists \phi$ that satisfies:

$$R_{Xt} = \phi(R_{Yt})$$

where R_{Xt} is the continuously compounded returns of the non marketed asset X.
R_{Yt} has a conditional normal distribution, and we can write [3] :

$$dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dZ$$

Using the latter theorem we know that Y returns are perfectly correlated with those of X. So, Y allows the identification of the non-marketed security uncertainty; X is well defined and we can use this process in order to solve optimal time problems.

1.3 Risk neutral valuation

This relation between the continuously compounded returns of X and Y is equivalent to the existence of a deterministic function ψ such that $X = \psi(Y)$. X is defined as a derivative on the virtual underlying Y, and the functional relation can be estimated using the constructed portfolio, as we will see in section 3. So X can be valued following derivative securities risk neutral valuation (Cox, Ross [4] and Harrison, Kreps [13]).

For this purpose we need Y risk-neutral process. Using standard arbitrage arguments, we find the market price of risk $\theta = \frac{\mu_Y - r}{\sigma_Y}$ and under the risk-neutral measure P^*, Y_t is given by the equation :

$$\begin{cases} dY_t = rY_t dt + \sigma_Y Y_t[\frac{\mu_Y - r}{\sigma_Y} dt + dZ] \\ = rY_t dt + \sigma_Y Y_t dZ^* \\ Y(0) = Y_0 \end{cases}$$

where Z^* is a brownian motion under $P*$, and r is the riskless continuously compounded interest rate (assumed to be constant). Now, the value of the derivative security X must be the expected value, under the risk-neutral measure, of its payoffs discounted to the valuation date at the risk-free rate.

We must now determine the function ϕ such that $R_{Xt} = \phi(R_{Yt})$. Existence of a deterministic function φ is characterized by $FCC = 1$.

2 Functional correlation coefficient

[3] $\sigma_Y dZ$ can be described using correlation matrix ρ and independent Brownian motions (Karatzas, Shreve [14]). This process parameters can be explicitly calculated

Suppose we observe n realizations of two random variables X and Y. We have a sample of n independent pairs (X_i, Y_i), distributed according to the joint distribution of (X, Y). The existence of any functional relation is given by a ratio of correlation (functional) of Y in X equal to 1. The FCC is defined by:

$$\eta^2_{Y|X} = \frac{\mathbf{V}\left(\mathbf{E}\left(Y|X\right)\right)}{\mathbf{V}\left(Y\right)}$$

FCC takes values in $[0,1]$, and $\eta^2_{Y|X} = 1$ is equivalent to the existence of a deterministic function f so that $Y = f\left(X\right)$ almost surely (Saporta [19]). To construct an empirical version of $\eta^2_{Y|X}$, in the case where X and Y are continuous, one only has to estimate, in a first step, conditional expectations i.e. regressions $g\left(X\right) \overset{def}{=} \mathbf{E}\left(Y|X\right)$. The construction proposed here uses a non-parametric method, whose principle is briefly described in appendix 1. It is based on the orthogonal functions method. From values of estimator $\widehat{g}_n$ at observed points X_i, we can define the empirical correlation ratio:

$$\eta^2_{emp} = \frac{\frac{1}{n}\sum_{i=1}^{n}\left(\hat{g_n}(X_i) - \bar{\hat{g}}\right)^2}{\frac{1}{n}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2}$$

where $\overline{\widehat{g}} = \frac{1}{n}\sum_{i=1}^{n}\hat{g}_n\left(X_i\right)$ et $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n}Y_i$

To answer to the problem of construction of an underlying asset, we suppose implicitly the existence of a functional relation between the geometric returns of X_0, given by R_{0t}, and the geometric returns of an "optimal "portfolio constituted of assets available on the market. Then we can construct portfolio Y whose returns are given by $R_Y = \sum_{j\in J}\lambda_j^* R_j$, in maximizing the correlation ratio between R_Y and R_0:

$$\eta^2_{R_Y|R_0} = \frac{\mathbf{V}\left(\mathbf{E}\left(R_Y|R_0\right)\right)}{\mathbf{V}\left(R_Y\right)}$$

It is written as a ratio of two quadratic functions in α_j:

$$\eta^2_{R_Y|R_0} = \frac{\sum_{j\in J}\sum_{k\in J}\lambda_j\lambda_k cov\left(\mathbf{E}\left(R_j|R_0\right), \mathbf{E}\left(R_k|R_0\right)\right)}{\sum_{j\in J}\sum_{k\in J}\lambda_j\lambda_k cov\left(R_j, R_k\right)}$$

In practice, optimal portfolio will be calculated from data by maximizing with regard to λ_j (under constraint $\sum_{j\in J}\lambda_j = 1$) empirical version of coefficient $\eta^2_{R_Y|R_0}$. Co-variances of numerator are valued from regressions estimated by the method describe above:

$$\eta^2_{emp} = \frac{\lambda\mathbf{M}_2\lambda' - \left(\lambda\mathbf{a}_2\right)^2}{\lambda\mathbf{M}_1\lambda' - \left(\lambda\mathbf{a}_1\right)^2} \tag{2.1}$$

with notations:

$\mathbf{M}_1 = \left(\frac{1}{T} \sum_{t=1}^{T} R_{jt} R_{kt} \right)_{j,k \in J}$
$\mathbf{M}_2 = \left(\frac{1}{T} \sum_{t=1}^{T} \widehat{g}_j (R_{0t}) \widehat{g}_k (R_{0t}) \right)_{j,k \in J}$
$\lambda = (\lambda_j)_{j \in J}$
$\mathbf{a}_1 = \left(\frac{1}{T} \sum_{t=1}^{T} R_{jt} \right)_{j \in J}$
$\mathbf{a}_2 = \left(\frac{1}{T} \sum_{t=1}^{T} \widehat{g}_j (R_{0t}) \right)_{j \in J}$
where $\widehat{g}_j$ is orthogonal functions estimator of regression $g_j (R_0) = \mathbf{E} (R_j | R_0)$.

With this optimal portfolio, the function ϕ so that $R_{0t} = \varphi (R_{Yt}) + \varepsilon_t$ can be estimated with the same non-parametric method.

In practice, we shall construct a portfolio of traded assets Y such there exists a deterministic function between R_0 and R_Y, by optimizing the FCC.

3 Empirical Procedure

The aim of this section is to frame and to establish reliability degree of the proposed optimization method. We construct a virtual underlying portfolio for financial real data and estimate the functional relation between returns and then between prices. We study problem of the portfolio composition stability over time.

3.1 Real financial data

We consider spot prices of N financial traded assets $\{X^1, ..., X^N\}$. We assume they follow Brownian processes:

$$\frac{dX^i}{X^i} = \mu_i dt + \sigma_i dZ_i$$

where μ_i are the instantaneous trends, σ_i are the standard deviations and dz_i are increments to Gauss-Wiener processes with correlation matrix ρ. We want to check our method on a traded asset X which plays the role of the asset to be valued.

Data are constituted by monthly returns of 30 financial assets of CAC 40, observed the first of each month, for which complete series are available. We work with the logarithm of these returns. Observation period is from 01/01/1989 to 01/09/1998, corresponding to a size samples of $T = 117$. We note R_{it} the geometric return of asset i at time t. For a fixed stock i_0, we want to determine, using the other 29 assets, the optimal portfolio whose geometric return is given by $R_{Y^*} = \sum_{j \neq i_o} \lambda_j^* R_j$, such that :

$$(\lambda_j^*)_{j \in J} = \max_{(\lambda_j)_{j \in J}} (\eta^2_{R_Y | R_{i_0}})$$

(with $\sum_{j\neq i_o}\lambda_j^*=1$).

These methods can be applied to i.i.d data, so a non-correlation test has been made for the 30 series. The values of empirical auto-correlation to the rank 20 and the corresponding values of Ljung-Box statistic show that the non-correlation hypothesis is not rejected at the threshold of 5% for 24 of the 30 series. We also test normality of series by the Bera Jarque statistic.

3.1.1 Optimization results

In a first step, we optimize the FCC, $\eta_{Y^*|R_{i_0}}$, between the geometric returns of an asset X_{i_0} and those of the portfolio. We obtain the optimal portfolio's coefficients $\left(\lambda_j^*\right)_{j\in J}$. We can also determine the deterministic function f such as $R_{Yt^*}=\hat{f}(R_{i_0t})+\epsilon_t$. But we want to estimate the function $\hat{\phi}$ such as $R_{i_0t}=\hat{\phi}(R_{Y^*t})+\epsilon_t$. So, the second step of this method consists in calculating the FCC $\eta_{R_{i_0}|R_{Y^*}}$, and estimate the functional relation $\hat{\phi}$ by the same non-parametric method.

In order to valuate this method we estimate the asset returns using the CAPM formula, and compare the quadratic errors of estimation obtained with these two methods.

We replicate all of the 30 assets available. To implement the maximization of the empirical coefficient, we used Matlab 5.2.0, with optimization Toolbox version 1.5.2. Note that during the application on real data, convergence of the algorithm was not reached in all cases.

Next table gives the quadratic error of estimation between the asset we want to replicate and the function of the optimal portfolio. The error using the FCC is given by :

$$EQ_\eta(i)=\frac{1}{T}\sum_{t=1}^{T}(R_{jt}-\hat{\phi}_j(R_{Y_t^*}))^2$$

where R_{Y^*} is the optimal portfolio that maximizes η^2_{emp} and $\hat{\phi}_j$ is the estimate regression function of R_j on R_{Y^*}.

The quadratic error for the CAPM is :

$$EQ_{CAPM}(j)=\frac{1}{T}\sum_{t=1}^{T}[R_{jt}-\beta_jR_{CAC,t}-r_0(1-\beta_j)]^2$$

where:

- β_j is the "bêta" of the asset j, that means sensitivity of the asset risk j to the CAC40 one;

- $(R_{CAC40,t})_{t=\overline{1,T}}$ is the CAC40 rates of return vector;

- $(1+r_0)^T=\prod_{t=1}^{T}(1+r_t)$, where r_t is the Banque de France monthly call for tender rate at time t ;

j	EQ_η	EQ_{CAPM}	j	EQ_η	EQ_{CAPM}
1	**0.0044**	0.0023	18	**0.0025**	0.0014
2	0.0016	0.0018	20	0.0022	0.0033
3	**0.0037**	0.0027	21	0.0058	0.0059
5	**0.0056**	0.0036	22	**0.0066**	0.0064
6	0.0036	0.0037	23	0.0018	0.0027
8	**0.0047**	0.0041	24	0.0014	0.0027
12	0.0022	0.0029	25	**0.0026**	0.0017
14	0.0041	0.0047	26	0.0015	0.0021
16	0.0019	0.0036	27	0.0016	0.0026
17	0.0022	0.0029	30	**0.0065**	0.0025

Mean Quadratic Errors

Mean quadratic error for FCC is often smaller than the CAPM one. And the two errors are of the same range.

We present graphic results of the optimal portfolio for two assets corresponding to $i_0 = 24$ (CCF) and $i_0 = 26$ (Vivendi), for which the η_{emp} is respectively 0.79 and 0.85. Next figures present, in each case, graph of geometric returns of $\phi(R_{Y^*})$ (dotted line) and R_{i_0} (solid line).

Application of this method on the same data set, with suppressing assets (six of them) having the highest partial auto-correlation, did not change the results significantly.

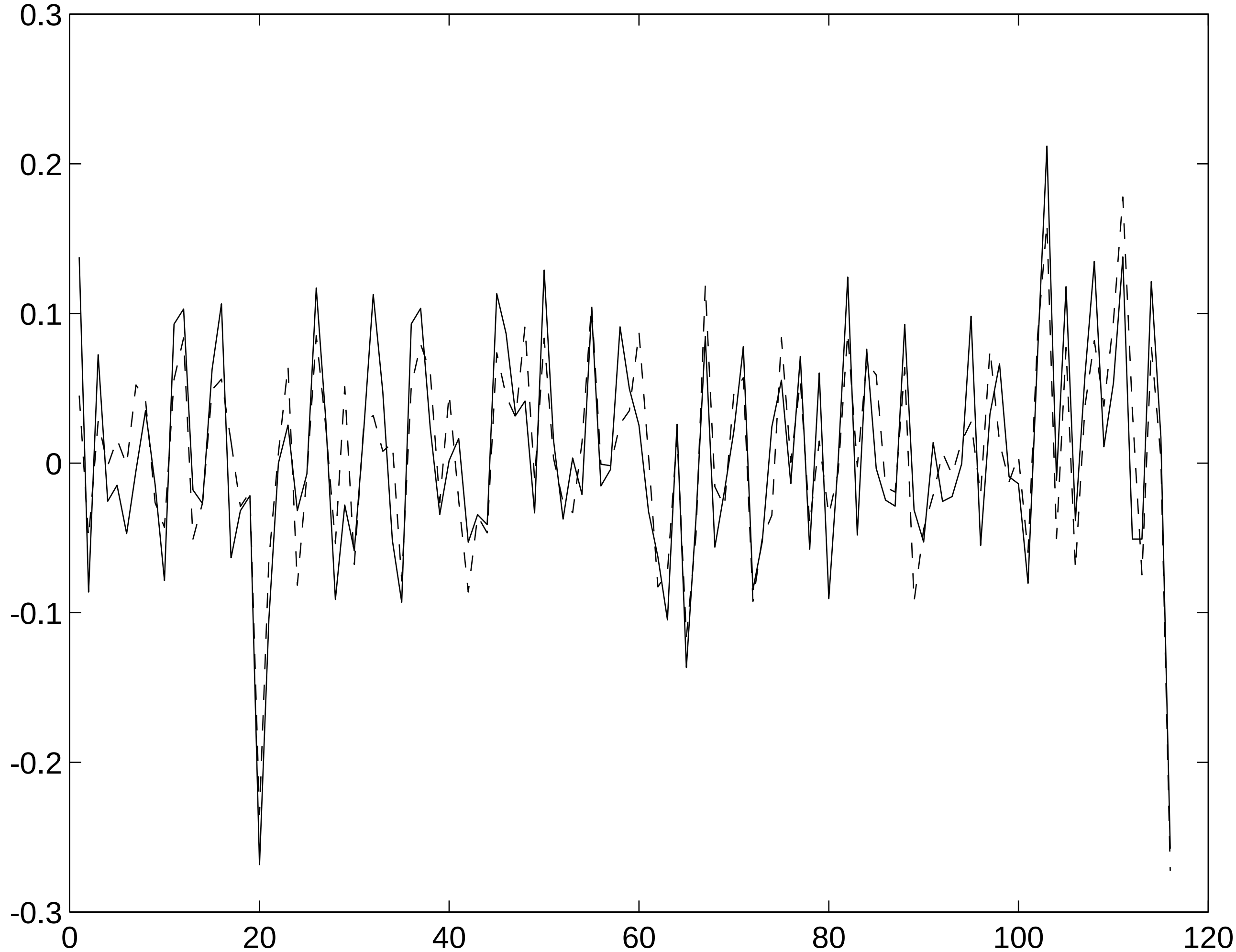

Figure 1: Comparison between $\phi(R_{Y^*})$ and R_{24}

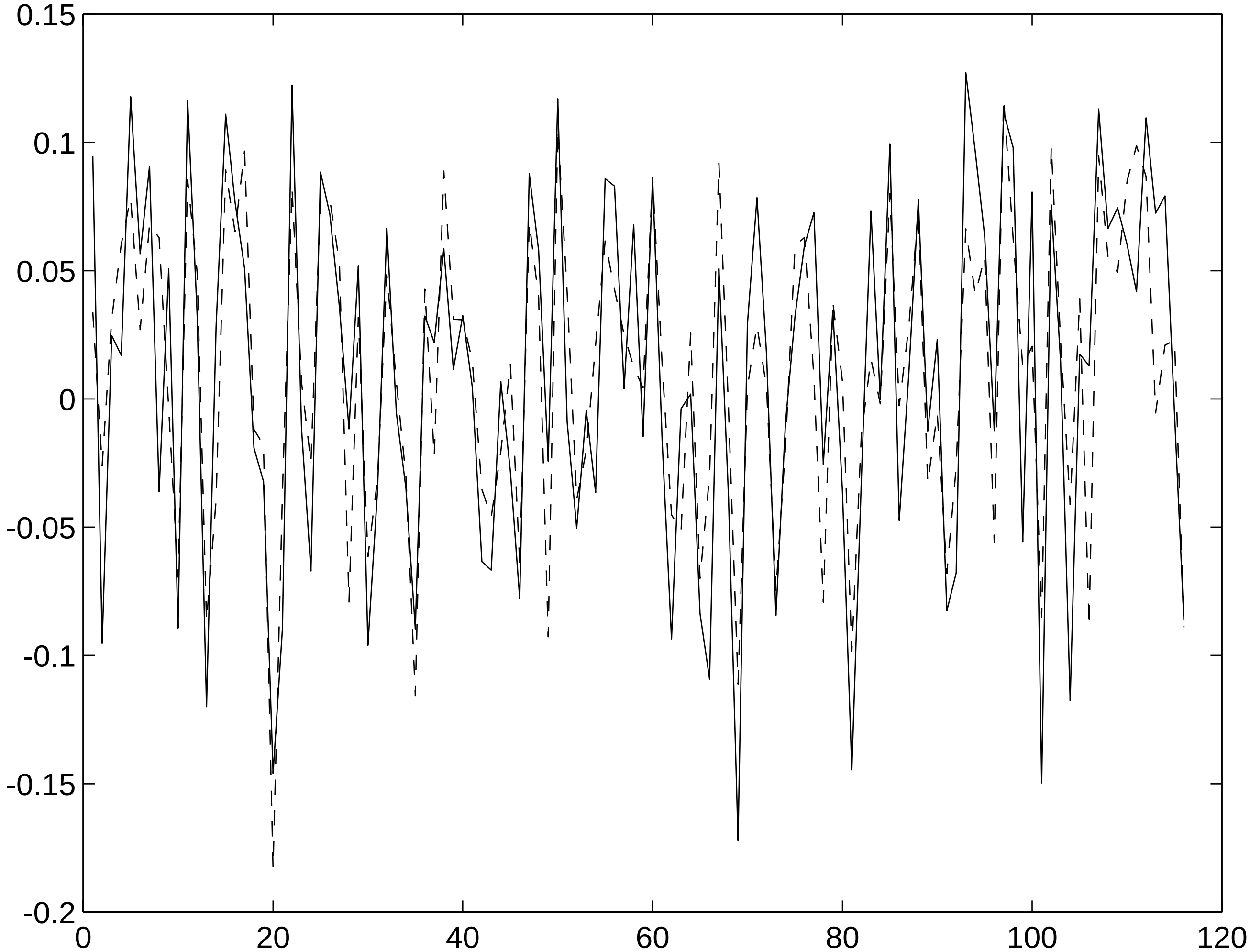

Figure 2: Comparison between $\phi(R_{Y^*})$ and R_{26}

3.1.2 Stability over time

We have tested the stability of the method over time as an indicator of its forecast power.

The initial period of observation $\{1,...,T\}$ has been divided in two sub-periods, the first for estimation and the second for forecast. Our method has been applied on data corresponding to the period $\{1,...,T-T_0\}$, with $T_0 \in \{6,\ 12,\ 24\}$. In this way, a number T_0 of months has been suppressed in each sample $(R_{jt})_j$ to calculate the optimal portfolio $\left(\lambda_j^*\right)_{j\in J}$. Then, forecasts of the portfolio rate of returns during period $\{T-T_0+1,...,T\}$ have been calculated according to the calculated coefficients λ_j^* and observed values R_{jt}. We compare this method with CAPM forecast, where the beta coefficients are estimated on the first period.

Next table gives the mean quadratic error of forecasts calculated with FCC and CAPM, for the two previous assets corresponding to $i_0 = 24$ and $i_0 = 26$, for horizon time $T_0 = 6, 12$ and 24 months.

	T_0	EQF_{FCC}	EQF_{CAPM}
	6	0.0028	0.0049
$i0 = 24$	12	0.0030	0.0046
	24	0.0050	0.0127
	6	0.00139	0.0050
$i0 = 26$	12	0.0064	0.0059
	24	0.0079	0.0065

Mean quadratic error of forecast

As for estimation error, the two methods forecast errors are comparable. But the FCC method is based on the functional relation between the asset we want to replicate and a portfolio. By construction of this portfolio, there is a relation and we can write it explicitly. On the contrary, the CAPM is based on the covariance between the asset and a market portfolio. In our example the market portfolio is the CAC 40, and the asset belongs to the CAC 40 's base. It is also obvious that the correlation with this index is different from zero and that some of the risk of the asset is explained by the market portfolio. In case we are interested in, where the asset is not a financial asset, this correlation is no more evident.

Moreover, the FCC method allows us to have a good fit of the asset process and we can deduce a relationship between prices.

3.1.3 Estimation of the functional relation between prices

In last sections we have constructed a portfolio $R_{Y^*} = \sum_i \lambda_i R_{X_i}$ such as $R_X = \phi(R_{Y^*})$. We can deduce that :

$$Y_t = \frac{Y_0}{\prod_{i=1}^{N} (X_0^i)^{\lambda_i}} \prod_{i=1}^{N} (X_t^i)^{\lambda_i}$$

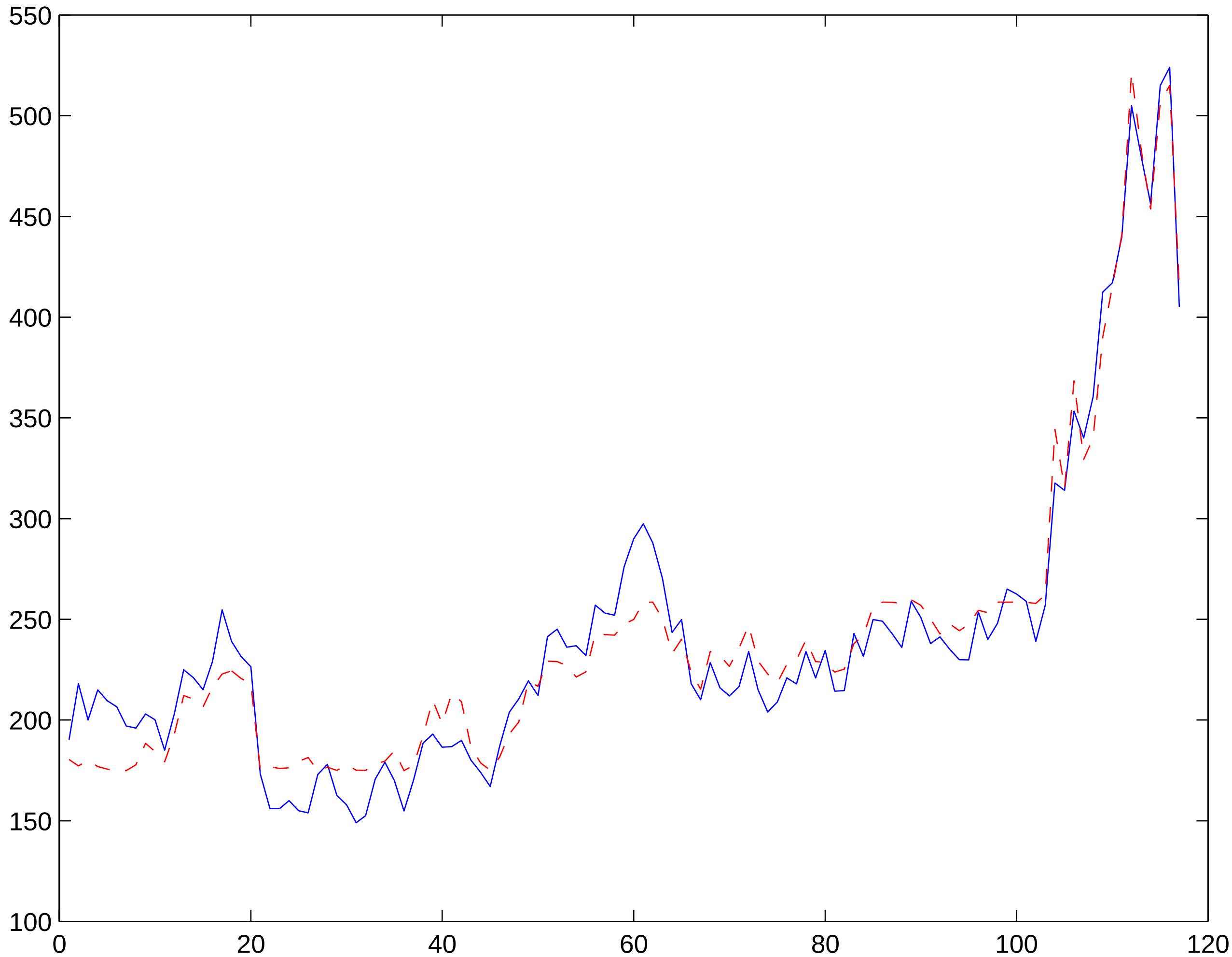

Figure 3: Comparison between $\hat{\psi}(Y^*)$ and X_{24}

we obtain prices of replication portfolio at each date, for every replicated asset. Using the same method of FCC, we can estimate a function $\hat{\psi}$ such as:

$$X_{i0} = \hat{\psi}(Y^*) + \epsilon$$

.

Next figures present results for the two previous assets, $i0 = 24$ and $i0 = 26$. We can observe that estimation of the prices are good fit of the real values, and values of $\eta^2_{Y|X}$ are respectively 0.9563 and 0.78.

This function will be use to solve valuation problem in the real options framework.

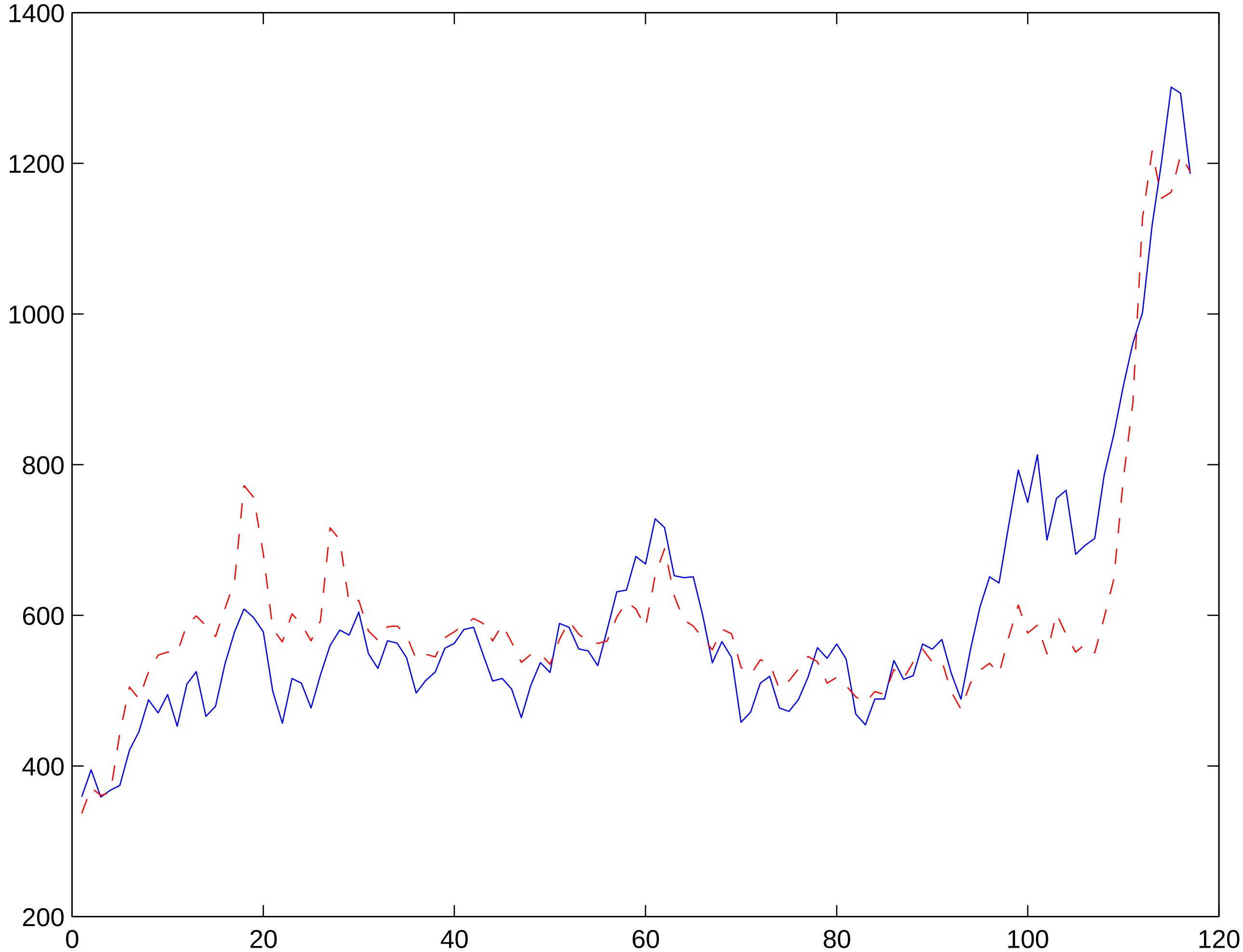

Figure 4: Comparison between $\hat{\psi}(Y^*)$ and X_{26}

Conclusion

Applications of real-options theory, beside standard case, come up against the question of choice of the traded underlying asset. To answer to this question, using an optimization procedure, we proposed a method based on statistical data. We have constructed a portfolio of traded assets perfectly correlated with the non marketed asset X, and such that X is defined as a derivative of the portfolio. This portfolio can be used for making a hedging strategy, short term forecasting or valuing the asset. To extend the result, we have to look at the sensitivity of the construction to the asset basis, and a way to select relevant assets in regard of the correlations. Our current work is to test this method on real data, more precisely on french unemployment allocation data.

Appendix 1

Proof:

$$\frac{X_{t+\Delta}}{X_t} = e^{\sigma_X\left(\tilde{Z}_{t+\Delta}-\tilde{Z}_t\right)-\frac{1}{2}\sigma_X^2\Delta+\mu_X\Delta}$$

$$= f\left(e^{\sigma_Y(Z_{t+\Delta}-Z_t)-\frac{1}{2}\sigma_Y^2\Delta+\mu_Y\Delta}\right)$$

$$\Rightarrow \tilde{Z}_{t+\Delta} - \tilde{Z}_t = \varphi(Z_{t+\Delta} - Z_t)$$

We can write[4]:

$$\tilde{Z}_{t+\Delta} - \tilde{Z}_t = \rho(Z_{t+\Delta} - Z_t) + \sqrt{(1-\rho^2)}(Z^{\perp}_{t+\Delta} - Z^{\perp}_t)$$

If $|\rho| \neq 1$,

$$\Rightarrow (Z^{\perp}_{t+\Delta} - Z^{\perp}_t) = \psi(Z_{t+\Delta} - Z_t)$$

$$E((Z^{\perp}_{t+\Delta} - Z^{\perp}_t)|(Z_{t+\Delta} - Z_t)) = const.$$

$$\Rightarrow \psi \equiv constant.$$

$$\Rightarrow contradiction$$

■

Appendix 2

Let $Z, \tilde{Z}$, two brownian motions with correlation $\tilde{\rho}$. We can write:

$$\tilde{Z}_t = \int_0^t \tilde{\rho}_s dZ_s + M_t$$

such as $M_t \perp Z$. So,

$$M_t = \int_0^t \alpha_s dZ^{\perp}_s.$$

$$t =< \tilde{Z}_t >= \int_0^t \tilde{\rho}_s^2 ds + < M >_t$$

$$= \int_0^t \tilde{\rho}_s^2 ds + \int_0^t \alpha_s^2 ds \Rightarrow \tilde{\rho}_s^2 + \alpha_s^2 = 1$$

i.e:

$$\tilde{Z}_t = \int_0^t \rho_s dZ_s + \int_0^t \sqrt{(1-\rho_s^2)} dZ^{\perp}_s$$

[4]see appendix 2

Appendix 3

Orthogonal functions method for regression estimation

We suppose regression function $g(x) \stackrel{def}{=} \mathbf{E}(Y|X=x)$ is square-integrable on support $\mathcal{X}$ of X and we will fix an orthonormed base $\left(e_j^n\right)_{j\geq 1}$ in $L^2(\mathcal{X})$. Then, g can be "decomposed" according to this base in a convergent series form, in the $L^2(\mathcal{X})$ meaning:

$$g = \sum_{j=1}^{\infty} c_j e_j, \text{ with } c_j = \int_{\mathcal{X}} g(x) e_j(x) dx.$$

The estimator of g is constructed from an rank q approximation of this series, $\sum_{j=1}^{q} c_j e_j$, whose coefficients are estimated from sample, by least squares method, i.e. by minimizing the quadratic error:

$$\left(\hat{c}_1^n, ..., \hat{c}_q^n\right)' = \arg\min_{a_j} \sum_{i=1}^{n} \left[Y_i - \sum_{j=1}^{q} a_j e_j(X_i)\right]^2.$$

From which esimator named of type "orthogonal functions series":

$$\hat{g}_n(x) = \sum_{j=1}^{q} \hat{c}_j^n e_j(X_i). \tag{3.1}$$

The number of terms q represents "smoothing parameter" of the method. In practice, it can be calculated from the sample by a cross-validation method.

As example of orthonormed bases on $\mathcal{X} = [-1, 1]$ [5], we can cite the base of Legendre polynomials or the one constituted by trigonometric functions (see for example Eubank [9] or Delecroix and Protopopescu [6] for more details concerning this method).

To correct brim effect due to periodic characteristic of trigonometric functions, one can wish to add to the series (3.1) some polynomial terms (see Eubank et Speckman [10]). The estimator obtained will be based on family of polynomial-trigonometric functions. In practice, it is sufficient to add two terms, the functions x and x^2.

References

[1] Black F., Scholes M. : The pricing of options and corporate liabilities, *Journal of Political Economy*, 81, pp 637-654, 1973.

[5] This choice of support $\mathcal{X}$ in not restrictive, because we can always do a one-to-one transformation of X_i to come back to a given interval.

[2] Brennan M., Schwartz E. : Evaluating natural ressource investments, *Journal of Business*, 58, pp135-157, 1985.

[3] Cortazar G., Schwartz E.S., Salinas M. :*Evaluating Environmental Investments : A Real Options Approach*, *Management Science*, 44:8, 1059-1070, 1998.

[4] Cox J., Ross S. : The valuation of options for alternative stochastic processes, *Journal of Financial Economics*, 3, 145-166.

[5] Cox J., Ross S., Rubinstein M. : Option pricing : a simplified approach, *Journal of Financial Economics*, 7, pp 229-263, 1979.

[6] Delecroix M., Protopopescu C. : Consistency of a Least Squares Orthogonal Series Estimator for a Regression Function, Discussion Paper No. 7, SFB 373, Humboldt-Universität zu Berlin, 1999.

[7] Dixit A., Pindyck R. : *Investment under uncertainty*, Princeton University Press, 1994.

[8] Duffie D. : *Dynamic Asset Pricing*, Princeton University Press, 1996

[9] Eubank R. L. : *Spline Smoothing and Nonparametric Regression*, Marcel Dekker, New York, 1988.

[10] Eubank R. L., Speckman P. : Convergence rates for trigonometric and polynomial-trigonometric regression estimators, *Statist. and Prob. Letters* **11** (1991), 119-124.

[11] Gibbons J. D. : *Non parametric Statistical Inference* Dekker, 1985.

[12] Gibson R. : *Option Valuation: Analyzing and pricing Standardized Option Contracts*, Geneve Georg Editor, 1988.

[13] Harrisson J., D. Kreps : Martingales and arbitrage in multiperiod securities markets, *Journal of Economic Theory*, 20, pp 381-408, 1979.

[14] Karatzas I., Shreve S.E. : *Brownian Motion and Stochastic Calculus*, Springer Verlag, New-York, 1991.

[15] Kast, R., Lapied A., Pardo S., Protopopescu C. : Evaluation des Risques Controversés par la Méthode des options Réelles, WP GREQAM dt-01B01, (*forthcoming in Economie et Prévision, 2001*).

[16] Kester W.C. : Today's Options for Tomorrow's Growth, *Harvard Business Review*, (March / April), pp 153-160, 1984.

[17] Merton R.: The theory of rational option pricing, *Bell Journal of Economics and Management Science*, 4, spring, pp. 141-183, 1973.

[18] Myers S.C. : Determinants of Corporate Borrowing, *Journal of Financial Economics*, 5, pp 147-175, 1977.

[19] Saporta G. : *Probabilités, Analyse de données et Statistique*, Technip, 1990.

[20] Shanken : The Arbitrage Pricing Theory : Is it testable?, *The Journal of Finance*, vol. 37, pp 1129-1140, 1982.

[21] Slade M. : Valuing managerial flexibility : An application of real options theory to mining investments, WP, University of British Coumbia, 2000.

[22] Trigeorgis L. : *Real options* , The MIT Press, Cambridge, 1996.