Risk Disaggregation As An Explanation Of The Smile : The Black & Scholes Formula Revisited.

Thierry Chauveau[¤] Hayette Gatfaoui[y]

September 2001

[¤] TEAM Pôle Finance (ESA 8059 du CNRS) of the University Paris I - Panthéon-Sorbonne, Maison des Sciences Economiques, 106-112, boulevard de L'Hôpital, 75013 Paris. Email : chauveau@univ-paris1.fr

[y] TEAM Pôle Finance (ESA 8059 du CNRS) of the University Paris I - Panthéon-Sorbonne, Maison des Sciences Economiques, 106-112, boulevard de L'Hôpital, 75013 Paris. Email : gatfaoui@univ-paris1.fr

Résumé

Dans leur formule, Black & Scholes évaluent le prix d'un call européen portant sur un actif sous-jacent sans distinction des risques qui composent ce dernier. En appliquant la méthode de pricing de Black & Scholes tout en faisant la distinction, à l'image de Sharpe (1964), entre le risque de marché et le risque idiosyncratique, nous obtenons une nouvelle formule de pricing de call européen. Les paramètres de cette formule comprennent les volatilités des deux facteurs de risque ou, de façon équivalente, la volatilité du facteur de marché ainsi que celle de l'action sous-jacente. Nous construisons alors un portefeuille destiné à dupliquer le facteur de marché, ce portefeuille de réplication étant un portfeuille diversi…é de façon naïve. Sous certaines conditions de régularité, l'e¤et de diversi…cation connu pour compenser les risques spéci…ques, s'applique. Le prix d'un call européen portant sur une action peut alors s'exprimer en termes des volatilités respectives du portefeuille de réplication et de l'action sous-jacente (ainsi que de son beta). Finalement, nous comparons notre formule à celle de Black & Scholes ainsi qu'à la méthode d'évaluation proposée par Corrado & Su. Nous mettons en évidence l'existence d'un smile de volatilité tout en fournissant une explication concurrente de celle proposée par les modèles à volatilité stochastique (i. e : Heston [1993]) ou par les modèles supposant une distribution non normale pour les rendements des actifs (i. e : Corrado & Su [1996, 1997]).

Mots clés : évaluation d'option, smile de volatilité, risque systématique, risque idiosyncratique, diversi…cation.

Abstract

In their formula, Black & Scholes evaluate a european call on an underlying asset without distinguishing between the di¤erent risks borne by the asset. Applying the Black & Scholes' pricing methodology and distinguishing between the market risk and idiosyncratic risk, as Sharpe [1964] did, we obtain a new pricing formula for a european call. The parameters of this formula include the volatilities of the two risk factors, or alternatively, the volatility of the market factor and that of the stock. We then build a market factor replicating portfolio (MFR) which is a naively divesi…ed portfolio. Under some regularity conditions, the diversi…cation e¤ect known to o¤set the speci…c risks applies. The price of a european call on a stock may then be expressed in terms of the volatilities of the MFR portfolio and of the underlying stock (and of its beta). Finally,we compare our formula to that of Black & Scholes and to the valuation proposed by Corrado & Su. We focus on the existence of a volatility smile and we give an explanation competing with the one proposed by stochastic volatility models (e. g. Heston [1993]) or models assuming a non normal distribution for the assets' returns (e. g. Corrado & Su [1996, 1997]).

Keywords : option pricing, volatility smile, systematic risk, idiosyncratic risk, diversi…cation.
JEL Codes : G11, G12, G13.

1 Introduction

The most famous concept of market equilibrium is that of the Capital Asset Pricing Model (henceforth CAPM). It was developed almost simultaneously by Sharpe [1963,1964], and Treynor [1961][1]. The major result given by the CAPM is that the total risk of an individual asset can be partitioned into two parts : systematic risk which is a measure of how the asset covaries with the economy and unsystematic risk which is independent of the economy. More precisely, any of the n traded risky assets exhibits an expected return which depends linearly on the level of a common factor –the market factor– and that of an independent idiosyncratic risk[2]. Such a point of view implies that, if time were allowed to vary explicitly[3], the dynamics of the n risky assets would be driven by that of $n + 1$ independent factors. Consequently, this type of model is inconsistent with the complete market hypothesis underlying the …nancial assets valuation …rst proposed by Black & Scholes [1973].

One could argue that such an inconsistency vanishes if one considers more sophisticated versions of the CAPM such as the one developped by Merton [1973] in which it is assumed that trading takes place continuously and that asset returns are distributed log-normally. If the risk-free rate of interest is non stochastic over time, the equilibrium returns must then satisfy an equation analogous to that of the elementary CAPM[4]. However, the dynamics of the n traded risky assets is determined by that of n independent factors. Hence, the market is complete and the return on the market portfolio is a linear combination of those of the n independent factors. The CAPM and Black and Scholes' valuation formula are compatible. Nevertheless, if the major result exhibited by the CAPM still holds when Merton's assumptions are made, its interpretation is very di¤erent from the original one. Strictly speaking, there is no longer any market factor.

One could also argue that, in later works dealing with arbitrage pricing, the complete market hypothesis has often be relaxed[5]. However, in almost all these studies, the market risk has still to be de…ned as a combination of individual risks.

To reconcile the "traditional view" –which we shall call Sharpe's approach– with the valuation proposed by Black & Scholes, we develop a two-factor model assuming that the price of any traded stock depends on the level of the market risk and of that of a speci…c risk; the levels of the two factors are assumed to be two independent geometric brownian motions. We then use such a risks' disaggregation for option pricing.

This paper is organized as follows. Using a complete …nancial market framework, we …rst address the issue of valuing an option whose underlying asset is

[1] Extensions of the model were provided by Mossin (1966), Lintner (1965,1969) and Black (1972) .

[2] There are thus as many speci…c risks as there are assets.

[3] Which is not ususally the case when elementary versions of the CAPM are under review.

[4] To spare space, we leave aside the case when the risk-free interest rate is stochastic and, consequently, when the model exhibits two risk factors (and, therefore, three-fund separation).

[5] For further details, see Cochrane & Saa-Requejo (1998) and Pham (1998).

a stock, given that the price of the stock is assumed to be determined by two independent risk factors, namely a market risk (common to all the stocks) and an idiosyncratic risk respectively[6]. It is easy to obtain an analytical formula for valuing european calls on stocks if it is further assumed that not only the n stocks but also the (n + 1) factors are traded. The same result is obtained under the weaker assumption that only the n stocks and the market factor are traded. In this case, the price of an option is no longer expressed in terms of the volatilities of the market factor and of the pertinent idiosyncratic factor but in terms of the volatility of the market factor and that of the underlying asset.

Second, we take into account the diversi…cation e¤ect to get rid of the unobservability of the market factor. If the number of assets is high enough –and particularly if it tends towards in…nity–, we can build a diversi…ed portfolio whose value is as close to that of the market portfolio as we want. The simultaneous observation of the portfolio's value and the prices of the n quoted assets is then equivalent to those of the n stocks and of the level of the market factor. In this case, the price of an option may be expressed in terms of the volatility of a naively diversi…ed market portfolio and that of the underlying asset. Option pricing remains possible even if none of the factors is tradable –which is the case in practice–.

Finally, we compare our pricing to that of Black & Scholes [1973] and of Corrado & Su ([1996, 1997]), using a simulation method. This framework allows us to focus on the existence of a volatility smile; we then give an explanation competing with the one proposed by stochastic volatility models (e. g. Heston [1993]) or models assuming a non normal distribution for the assets' returns (e. g. Corrado & Su [1996, 1997]).

2 Option pricing with two factors : reconciling Sharpe's and Black & Scholes' points of view.

In this section, we use the framework of Black & Scholes [1973] except that the price of the underlying asset does no more depend on a unique risk factor but on two risk factors.

Assumptions :

We consider the price S_t of a stock : it is an F_t-adapted process where F_t represents the available information contained in the stock's price at the current date t. We suppose that the dynamic of S_t depends both on a market risk factor whose level is labelled X_t and on an idiosyncratic risk factor whose value is denoted I_t. The former represents a risk associated to the economic

[6] The complete market hypothesis is equivalent to assume that the two risk factors are perfectly observable.

state and/or to the business cycle. The latter represents a liquidity risk and/or a default risk[7]. Finally, we define α and β as being two fixed deterministic constants. The main assumption is that the price S_t depends [8] on the two risk factors, according to the following formula:

$$S_t = \alpha X_t^{\beta} I_t \tag{1}$$

We further assume that the levels of the risk factors follow Ito processes such that:

$$dX_t = X_t \left(\mu(t; X_t)dt + \sigma(t; X_t)dW_t \right) \tag{2}$$

$$dI_t = I_t \left(\mu^{\alpha}(t; I_t)dt + \sigma^{\alpha}(t; I_t)dW_t^{\alpha} \right) \tag{3}$$

where

- W_t and W_t^{α} are two independant brownian motions[9] under the historical probability;
- $\mu(t; X_t); \sigma(t; X_t); \mu^{\alpha}(t; I_t)$ and $\sigma^{\alpha}(t; I_t)$ are continuous real valued functions (on $R^+ \times R$) which are one time differentiable relatively to the time and two times differentiable relatively to their second argument. They are supposed to satisfy the appropriate Lipschitz conditions to assure the unicity of the solutions of their respective stochastic differential equations (SDE) (conditionally to a fixed starting value on the considered time horizon)

Note that Equations (1), (2) and (3) imply that we have :

$$R_{X_t} = d \ln X_t = \left(\mu(t; X_t) - \frac{\sigma^2(t; X_t)}{2} \right) dt + \sigma(t; X_t)dW_t \tag{4}$$

$$R_{I_t} = d \ln I_t = \left(\mu^{\alpha}(t; I_t) - \frac{\sigma^{\alpha 2}(t; I_t)}{2} \right) dt + \sigma^{\alpha}(t; I_t)dW_t^{\alpha} \tag{5}$$

and, consequently :

[7] This decomposition follows the point of view of Wilson (1998) because it encompasses both the systematic part (induced by the economic state and the interest rates) and the specific part (peculiar to each debt issuer) of the credit risk.

[8] This specification implicitly supposes that the following conditions are satisfied : $\alpha \geq 0$ and $\forall t \geq 0 \quad X_t \geq 0; I_t \geq 0$.

[9] The market risk and idiosyncratic risk factors are independant. In this case, we have $F_t = \sigma(S_u; 0 \leq u \leq t) = \sigma(W_u; 0 \leq u \leq t) \vee \sigma(W_u^{\alpha}; 0 \leq u \leq t)$.

$$R_{S_t} = \beta R_{X_t} + R_{I_t}$$

where R_{S_t}, R_{X_t} and R_{I_t} are the returns on the contingent claims whose value are $S_t; X_t$ and I_t. According to Sharpe, R_{I_t} is the sum of the risk-free rate of interest R_{f_t} and of a random variable whose expected value is zero ($R_{I_t} = R_{f_t} + \rho_{I_t}$ with $E\left[\rho_{I_t} | F_t\right] = 0$). Finally, equations (4) and (5) are coherent with Sharpe's view if we identify $\mu^{\ast}(t; I_t) - \frac{\sigma^{\ast 2}(t; I_t)}{2}$ to $(1 - \beta) R_{f_t}$. We now study the influence of the assumptions above-mentioned on the stock's dynamic and then on the valuation of a call written on this stock.

Dynamic of the price of the underlying asset :

We apply Ito's lemma to the underlying asset's price considered as a two-variables function $S_t = S(X_t; I_t)$. Indeed, S_t depends exclusively on the levels of the market and of the specific risks[10] :

$$dS_t = \frac{\partial S}{\partial X} dX_t + \frac{\partial S}{\partial I} dI_t + \frac{1}{2}\left(\frac{\partial^2 S}{\partial X^2}\sigma^2(t; S_t) X_t^2 + \frac{\partial^2 S}{\partial I^2}\sigma^{\ast 2}(t; I_t) I_t^2\right) dt$$

We replace dX_t and dI_t with their respective expressions and we compute the partial derivatives of price of the stock with respect to the risk factors :

$$\frac{\partial S}{\partial X} = \beta \ast X_t^{\beta - 1} I_t = \beta \frac{S_t}{X_t} \qquad \frac{\partial S}{\partial I} = \ast X_t^{\beta} = \frac{S_t}{I_t}$$

$$\frac{\partial^2 S}{\partial X^2} = \beta(\beta - 1) \ast X_t^{\beta - 2} I_t = \beta(\beta - 1)\frac{S_t}{X_t^2} \qquad \frac{\partial^2 S}{\partial I^2} = 0$$

which leads to the following relation :

$$\frac{dS_t}{S_t} = \Upsilon(t; X_t; I_t) dt + \Psi(t; X_t; I_t) d\omega_t$$

with

$$\Upsilon(t; X_t; I_t) = \beta\mu(t; X_t) + \mu^{\ast}(t; I_t) + \frac{1}{2}\beta(\beta - 1)\sigma^2(t; X_t)$$

$$\Psi(t; X_t; I_t) = \sqrt{\beta^2\sigma^2(t; X_t) + \sigma^{\ast 2}(t; I_t)}$$

$$d\omega_t = \frac{\beta\sigma(t; X_t) dW_t + \sigma^{\ast}(t; I_t) dW_t^{\ast}}{\sqrt{\beta^2\sigma^2(t; X_t) + \sigma^{\ast 2}(t; I_t)}} \text{ : standard brownian motion}$$

[10] S is a continuous and twice differentiable function with respect to each argument X and I.

We obtain the stock's dynamic expressed in terms of the parameters characterizing the evolution of the risk factors encompassed in its price. Note that the existence of this SDE's solution results from the continuity of the parameters characterizing the diffusions of the two risk factors. The uniqueness results from the Lipschitz conditions which have been assumed to hold.

Dynamic of the call's price :

We now consider a european call written on the stock, namely the underlying asset, with a strike price K and a maturity date T. The value C of the european call depends on the levels of the two risk factors and on time; it can be specified as follows :

$$C = C(t, X_t, I_t)$$

which allows us to apply the multivariate Ito's lemma to the call's price [11]. We then obtain the following equation :

$$\begin{aligned} dC = & \left[\begin{array}{c} C_X \mu(t, X_t) X_t + C_I \mu^*(t, I_t) I_t + C_t \\ + \frac{1}{2} \left(C_{XX} \sigma^2(t, X_t) X_t^2 + C_{II} \sigma^{*2}(t, I_t) I_t^2 \right) \end{array} \right] dt \\ & + \left[C_X \sigma(t, X_t) X_t dW_t + C_I \sigma^*(t, I_t) I_t dW_t^* \right] \end{aligned}$$

under the limit condition :

$$C(T, X_T, I_T) = (S_T - K)^+ = Max(0, S_T - K)$$

As usual, we first consider a self-financing strategy allowing the replication of the call with the two risk factors. The value V_t of the replicating portfolio, is defined by

$$V_t = \Delta_t I_t + \Gamma_t X_t = C(t, X_t, I_t)$$

which gives $\Delta_t = C_I$ and $\Gamma_t = C_X$.

[11] We use the following standard denominations::

$$\frac{\partial C}{\partial t} = C_t \quad \frac{\partial^2 C}{\partial X \partial I} = \frac{\partial^2 C}{\partial I \partial X} = C_{XI} = C_{IX}$$

$$\frac{\partial C}{\partial X} = C_X \quad \frac{\partial^2 C}{\partial X^2} = C_{XX}$$

$$\frac{\partial C}{\partial I} = C_I \quad \frac{\partial^2 C}{\partial I^2} = C_{II}$$

$$C_Z = \begin{bmatrix} C_X \\ C_I \end{bmatrix} \quad C_{ZZ} = \begin{bmatrix} C_{XX} & C_{XI} \\ C_{IX} & C_{II} \end{bmatrix}$$

Then, we build a hedging portfolio, using the replicating portfolio and the call itself[12]. The hedging portfolio is immune against the market risk and the idiosyncratic risk, and its value π_t may be expressed as :

$$\pi_t = V_t - C(t; X_t; I_t) = C_I I_t + C_X X_t - C(t; X_t; I_t)$$

If there is no arbitrage opportunity, the return of the immune portfolio is equal to the risk free rate of return[13]; hence:

$$d\pi_t = r\pi_t dt$$

$$C_I dI_t + C_X dX_t - dC = r\left(C_I I_t + C_X X_t - C(t; X_t; I_t)\right) dt$$

Replacing the stochastic differentials with their respective expressions and grouping the stochastic and the deterministic terms, we get the following partial differential equation (PDE) :

$$rC_X X_t + \frac{1}{2}C_{XX}\sigma^2(t; X_t)X_t^2 + C_t - rC + \frac{1}{2}C_{II}\sigma^{*2}(t; I_t)I_t^2 + rC_I I_t = 0$$

We now assume that the level of each risk factor follows a geometric brownian motion with constant drift and volatility[14], that is to say :

$$\mu(t; X_t) = \mu \qquad \sigma(t; X_t) = \sigma$$
$$\mu^*(t; I_t) = \mu^* \qquad \sigma^*(t; I_t) = \sigma^*$$

where $\mu; \sigma; \mu^*$ and σ^* are deterministic constants. We therefore assume a constant risk-free rate of interest ($\mu^* - (\sigma^{*2}/2) = R_f$.). The PDE above-mentioned takes the form :

$$\left[rC_X X_t + \frac{1}{2}C_{XX}\sigma^2 X_t^2 + C_t - rC\right] + \left[\frac{1}{2}C_{II}\sigma^{*2}I_t^2 + rC_I I_t\right] = 0$$

PDE resolution :

We show in the appendix that the PDE resolution leads to the following formulation for the price of the two risk factors european call :

$$C(T - t; X_t; I_t; K; r; \sigma; \sigma^*) \tag{6}$$
$$= \gamma_t \int_{\mathbb{R}} \frac{e^{-\frac{1}{2}w^2}}{\sqrt{2\pi}} N\left(\tilde{d}_1(w)\right) dw - Ke^{-r(T-t)} \int_{\mathbb{R}} \frac{e^{-\frac{1}{2(T-t)}\frac{w^2}{\sigma^2}}}{\sigma\sqrt{2\pi(T-t)}} N\left(\tilde{d}_2(w)\right) dw$$

---

[12] The hedging portfolio corresponds to the selling of one unit of call and to the simultaneous buying of Δ_t units of specific risk factor and of Γ_t units of market factor.

[13] This interest rate is temporarily supposed constant.

[14] This is equivalent to suppose that each factor's rate of return follows a normal distribution law.

with

$$\gamma_t = \Lambda X_t^{\beta} I_t\, e^{\left[\beta\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{2}\beta^2\sigma^2\right](T-t)} = S_t\, e^{\left[\beta\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{2}\beta^2\sigma^2\right](T-t)}$$

$$\widehat{d}_1(w) = \frac{\ln\left(\frac{\Lambda X_t^{\beta} I_t}{K}\right) + w\beta\sigma\sqrt{(T-t)} + \left[\begin{array}{c}\beta\left(r-\frac{\sigma^2}{2}\right)+\beta^2\sigma^2 \\ +\left(r+\frac{\sigma^{*2}}{2}\right)\end{array}\right](T-t)}{\sigma^*\sqrt{(T-t)}}$$

$$\widehat{d}_2(w) = \frac{\ln\left(\frac{\Lambda X_t^{\beta} I_t}{K}\right) + \beta w + \left[\beta\left(r-\frac{\sigma^2}{2}\right) + \left(r-\frac{\sigma^{*2}}{2}\right)\right](T-t)}{\sigma^*\sqrt{(T-t)}}$$

For further details about the computation of the analytical formula associated to the pricing of the european call, the reader is invited to consult the appendix. Note that the case $\beta = 0$ corresponds to the formula of Black & Scholes with $S_t = \Lambda I_t$. Moreover recalling that $\Psi = \sqrt{\sigma^{*2} + \sigma^2\beta^2}$, the formula above-mentioned has the new expression :

$$C(T-t; S_t; K; r; \Psi; \sigma) \qquad (7)$$

$$= \gamma_t \int_{\mathbb{R}} \frac{e^{-\frac{1}{2}w^2}}{\sqrt{2\pi}} N\left(\widehat{d}_1(w)\right) dw - K e^{-r(T-t)} \int_{\mathbb{R}} \frac{e^{-\frac{1}{2(T-t)}\frac{w^2}{\sigma^2}}}{\sigma\sqrt{2\pi(T-t)}} N\left(\widehat{d}_2(w)\right) dw$$

with

$$\gamma_t = S_t\, e^{\left[\beta\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{2}\beta^2\sigma^2\right](T-t)} = \gamma(S_t; K; r; (T-t); \Psi; \sigma; \beta)$$

$$\widehat{d}_1(w) = \frac{\ln\left(\frac{S_t}{K}\right) + w\beta\sigma\sqrt{(T-t)} + \left[\beta\left(r-\frac{\sigma^2}{2}\right) + \frac{\beta^2\sigma^2}{2} + \left(r+\frac{\Psi^2}{2}\right)\right](T-t)}{\sqrt{(T-t)\left(\Psi^2-\beta^2\sigma^2\right)}}$$

$$\widehat{d}_2(w) = \frac{\ln\left(\frac{S_t}{K}\right) + \beta w + \left[\beta\left(r-\frac{\sigma^2}{2}\right) + \frac{\beta^2\sigma^2}{2} + \left(r-\frac{\Psi^2}{2}\right)\right](T-t)}{\sqrt{(T-t)\left(\Psi^2-\beta^2\sigma^2\right)}}$$

We then notice that the option's price depends both on the market factor and the price of the traded asset. The valuation remains therefore possible when the specific factors become unobservable. Indeed, we no more need to know the volatilities σ and σ^* of the market and idiosyncratic factors respectively but only the volatilities σ and Ψ of the market factor and the underlying respectively.

3 Diversification or indirect observation of the market factor

To solve the problem posed by the unobservability of the risk factors–which is the case in practice–, we adapt the Capital Asset Pricing Model (CAPM) to the framework used in the preceding section. We assume the existence of a financial market where of $(n+1)$ assets are traded (n risky assets and one risk free asset). The n first assets may be viewed as stocks and the safe asset as a bill.

Notations and analytical framework :

We assume that all the assumptions made in Section 2 still hold. We define S_t^i as the value of the i th stock at the current date t. From Section 2, we know that the prices obey the following dynamic :

$$S_t^i = \ast X_t^{\beta_i} I_t^i$$

$$\text{with } \forall 0 \leq i \leq n; \forall t \geq 0; \quad \begin{cases} dX_t = X_t(\mu dt + \sigma dW_t) \\ dI_t^i = I_t^i \mu_i^\ast dt + \sigma_i^\ast dW_{t;i}^\ast \end{cases}$$

where $\forall 0 \leq i \leq n; \forall t \geq 0; \quad \mu; \sigma; \mu_i^\ast$ et $\sigma_i^\ast$ are some constant values. In this case, we know that the dynamic of each risky asset i in the historical universe has the form :

$$dS_t^i = S_t^i \left(\beta_i \mu + \mu_i^\ast + \frac{1}{2}\beta_i(\beta_i - 1)\sigma^2 \right) dt + S_t^i \left[\beta_i \sigma dW_t + \sigma_i^\ast dW_{t;i}^\ast \right]$$

which transforms, in the risk neutral universe, as :

$$dS_t^i = S_t^i \left(\frac{1}{2}\beta_i(\beta_i - 1)\sigma^2 + r(\beta_i + 1) \right) dt + S_t^i \left[\beta_i \sigma d\overline{W}_t + \sigma_i^\ast d\overline{W}_{t;i}^\ast \right]$$

We then build a diversified portfolio, i.e. including all the risky assets. The proportion of security i in the portfolio is denominated a_t^i and the value M_t of the portfolio[15] is hence :

$$\forall t \geq 0; \quad M_t = \sum_{i=1}^{n} a_t^i S_t^i \quad \forall 0 \leq i \leq n; \quad a_t^i \in \mathbb{R} \quad \text{with} \sum_{i=1}^{n} a_t^i = 1$$

Diversification effect :

From now on, we consider a naively diversified portfolio, namely a uniformly weighted portfolio. For reasons which will become apparent later, we shall call it the market-factor replicating portfolio (MFR portfolio). Hence, we have :

[15] The weights $\left(a_t^i\right)$ of the market portfolio are therefore bounded on the space of the real numbers (since $\forall i; |n_i| \leq N$ implies $\left|a_t^i\right| \leq 1$).

$$\alpha_t^i = \frac{a_t^i S_t^i}{M_t} = \frac{1}{n}$$

and the dynamic of the value of the MFR portfolio then reads :

$$\frac{dM_t}{M_t} = \bar{\Theta} dt + \bar{\beta}\sigma d\overline{W}_t + \frac{1}{n}\left(\sum_{i=1}^{n} \sigma_i^* d\overline{W}^*_{t;i}\right)$$

where

$$\bar{\Theta} = \left(\tfrac{1}{n}\sum_{i=1}^{n} \Theta_i\right) \qquad \bar{\beta} = \left(\tfrac{1}{n}\sum_{i=1}^{n} \beta_i\right)$$

Conditionally to the information[16] available at time t and under the risk neutral martingale measure, we can express the total risk associated to the MFR portfolio's return as:

$$Var\left(\frac{dM_t}{M_t}\right) = \left(\bar{\beta}_t^2\sigma^2 + \sum_{i=1}^{n} \alpha_t^{i2}\sigma_i^{*2}\right)dt$$

or, equivalently :

$$Var\left(\frac{dM_t}{M_t}\right) = \left(\bar{\beta}^2\sigma^2 + \frac{\overline{\sigma^{*2}}}{n}\right)dt$$

where $\overline{\sigma^{*2}}$ is the average idiosyncratic risk ($\overline{\sigma^{*2}} = \frac{1}{n}\left(\sum_{i=1}^{n} \sigma_i^{*2}\right)$). Consequently, we can conclude that, when the number n of securities tends towards infinity, we have :

$$Var\left(\frac{dM_t}{M_t}\right) = \bar{\beta}^2\sigma^2 dt$$

The diversification effect tends to offset the specific risk of the MFR portfolio. The risk borne by the uniformly weighted market portfolio depends only on its systematic risk[17] and the volatility σ_M of the MFR portfolio,when n is infinitely high, then reduces to the following expression :

$$\sigma_M = \bar{\beta}\sigma$$

[16] Note that, by definition, α_i is a F_t-adapted process whatever i under a filtration F_t defined as following : $F_t = \bigcup_{i=1}^{n} \sigma\left(W^*_{s;i}; 0 \le s \le T\right) \cup \sigma(W_s; 0 \le s \le T)$. Analogously, the processes S_t^i, M_t, σ, σ_i^* and Θ_i are F_t-adapted. Moreover, under this filtration F_t, the risk neutral measure is such that $\overline{W}_t$ and $\forall i \in \{1; \ldots; n\}$; $\overline{W}^*_{t;i}$ are independant standard brownian motions.

[17] This is coherent with the view of Wilson (1998) arguing that the specific part of credit risk could be diversified.

Moreover, the price of the european call on stock i now reads, using the same denominations as before, except that the superscript i is now explicitly taken into account :

$$C\left(T-t; S_t^i; K^i; r; \alpha_i; \frac{\sigma_M}{\bar\beta}; \beta_i\right) \tag{8}$$

$$= \gamma_t^i \int_R \frac{e^{-\frac{1}{2}w^2}}{\sqrt{2\pi}} N\left(\hat d_1(w)\right) dw - K^i e^{-r(T-t)} \int_R \frac{e^{-\frac{1}{2(T-t)}\frac{w^2}{\sigma^2}}}{\sigma\sqrt{2\pi(T-t)}} N\left(\hat d_2(w)\right) dw$$

with

$$\gamma_t^i = S_t^i\, e^{\left[\beta_i\left(r-\frac{(\sigma_M/\bar\beta)^2}{2}\right)+\frac{1}{2}\left(\frac{\beta_i\sigma_M}{\bar\beta}\right)^2\right](T-t)}$$

$$\hat d_1(w) = \frac{\ln\left(\frac{S_t^i}{K^i}\right) + w\left(\frac{\beta_i\sigma_M}{\bar\beta}\right)\sqrt{(T-t)} + \left[\beta_i\left(r-\frac{(\sigma_M/\bar\beta)^2}{2}\right) + \frac{(\beta_i\sigma_M/\bar\beta)^2}{2} + r + \frac{\alpha_i^2}{2}\right](T-t)}{\sqrt{(T-t)\left[\alpha_i^2-(\beta_i\sigma_M/\bar\beta)^2\right]}}$$

$$\hat d_2(w) = \frac{\ln\left(\frac{S_t^i}{K^i}\right) + \beta_i w + \left[\beta_i\left(r-\frac{(\sigma_M/\bar\beta)^2}{2}\right) + \frac{(\beta_i\sigma_M/\bar\beta)^2}{2} + r - \frac{\alpha_i^2}{2}\right](T-t)}{\sqrt{(T-t)\left[\alpha_i^2-(\beta_i\sigma_M/\bar\beta)^2\right]}}$$

Finally, on a financial market where n risky assets and one risk free asset are traded in such a way that each risky asset depends on a market factor and an idiosyncratic factor according to equation (6), the factors' unobservability does not preclude the risk neutral valuation of options. One has just to substitute the ratio $\frac{\sigma_M}{\bar\beta}$ of the volatility of the market portfolio's return to the average beta for the volatility σ of the market factor. The price of an option may be expressed in terms of the volatilities of a naively diversified market portfolio and of the underlying asset respectively. Option pricing remains therefore possible even if none of the factors is tradable –which is the case in practice–.

4 Comparison with the formulae of Black & Scholes and of Corrado & Su

In this section, we compare our option valuation formula to the valuations proposed by Black & Scholes (1973) and by Corrado & Su (1996, 1997). Given the complexity related to the computation of the comparative statics of our analytical formula, the comparisons are achieved through simulations.

4.1 Black & Scholes

4.1.1 Comparison with a varying beta

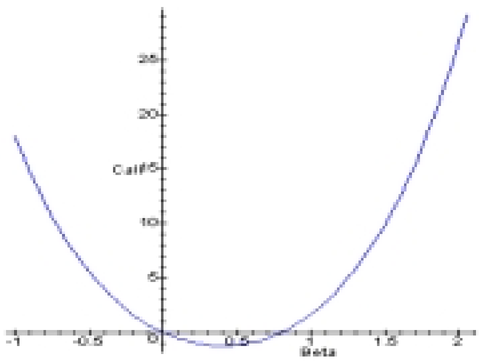
In this subsection, we carry out simulations where beta is the only parameter which is varying. Simulations are achieved using the next values of parameters :

$$S = K = 100 \quad ¾ = ¾^{¤} = 1$$
$$r = 0{:}10 \quad (T \ ¡ \ t) = 0{:}25$$

We therefore consider at-the-money calls and, to make the comparison fair, we impose the next relation:

$$¾_{BS}^2 = {}^{-2}\, ¾^2 + ¾^{¤2}$$

where $¾_{BS}$ represents the aggregated volatility of the underlying (i. e. without any distinction between speci…c and systematic risks). We then plot the pricing di¤erence between the call prices induced from our two factors valuation method and those issued from the Black & Scholes' valuation in terms of the beta parameter characterizing the underlying, namely the di¤erence between the two factors pricing and the Black & Scholes' one :



Di¤erence between the two factors pricing and Black & Scholes' pricing.

We observe that the pricing di¤erence grows when the beta increases, and reduces when the beta starts decreasing from the level of 0:40. Notice that, as expected, the pricing error is null for a zero beta since for a null beta our valuation formula reduces to the Black & Scholes' formula. Moreover, the pricing di¤erence is also null when the beta is equal to 0:80.

4.1.2 The role of “moneyness”

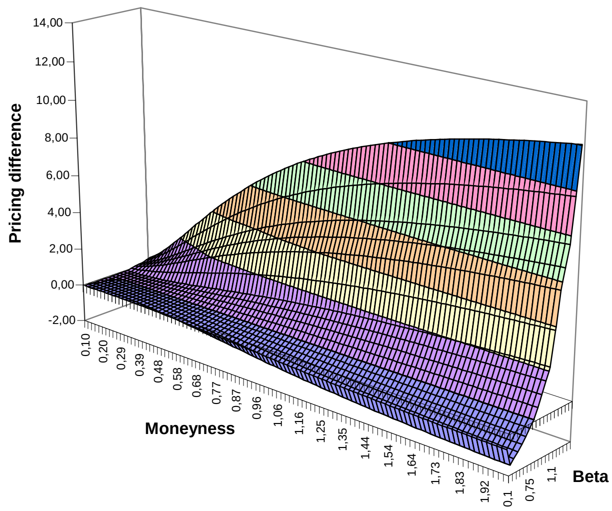
We now express the prices of calls as functions of the moneyness (i. e. ratio of the underlying’s price to the strike (S/K)) associated to the pricing, given a fixed beta. Simulations are undertaken, using the following values for the other parameters :

$$\begin{array}{ll} K = 100 & \sigma = \sigma^{*} = 1 \\ S = \text{moneyness} \times K & \\ r = 0.10 & (T - t) = 0.25 \end{array}$$

We still have the following relation :

$$\sigma_{BS}^2 = \beta^2 \sigma^2 + \sigma^{*2}$$

We successively realize our simulations for some values of the beta[18] ranging from 0.1 to 1.5. To have a global view, we plot the european call pricing difference between our two factors method and the valuation of Black & Scholes in function of the moneyness and of the beta. The graph under-mentioned illustrates then the pricing difference between our two factors methodology and the valuation of Black & Scholes for varying beta and moneyness parameters.



Difference between the european call prices.

Note that for beta values inferior to 0.8, the pricing difference is negative (i. e. : the pricing of Black & Scholes overestimates the european call price) while in the opposite case, this difference is positive (i. e. : the pricing of Black &

[18] The case $\beta = 0$ is uninteresting because our two factors pricing formula reduces to the valuation formula of Black & Scholes, which gives a nul pricing error whatever the value of the moneyness.

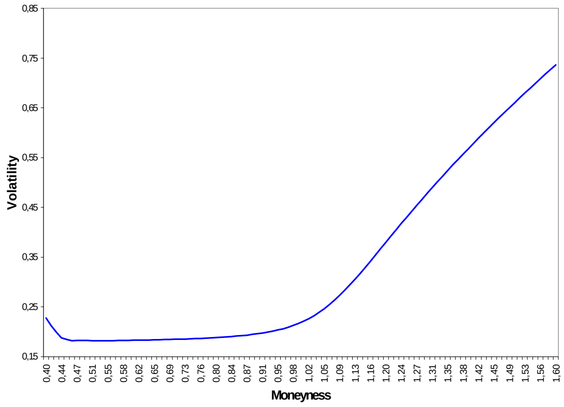
Scholes underestimates the european call price). This di¤erence being almost null when the beta equals 0:8. Moreover, the more the moneyness is high, the more important is the pricing di¤erence in absolute value.

4.1.3 Existence of a volatility smile

We consider here that call prices induced from our two factors formula (i. e. : the pricing of Chauveau & Gatfaoui) correspond to the prices observed on the market. We assume the following values of parameters :

K = 100	¾ = ¾¤ = 0:16
S = moneyness ¤ K	¯ = 0:5
r = 0:10	(T ¡ t) = 0:25

Starting from the call prices above-mentioned, we invert the Black & Scholes' formula to deduce the associated values of the implied volatility, which allows to plot the graph corresponding to the implied volatility's evolution in function of the moneyness.



We then …nd again the volatility smile describing the well known bias of the Black & Scholes' formula generated by the constant volatility hypothesis. Besides, this bias is the object of a correction method recently proposed and which is presented in the next section.

4.2 Corrado & Su

In this section, we compare our pricing formula with the ones of Corrado & Su and of Black & Scholes respectively in order to realize some comparisons between those three valuation methods.

4.2.1 Pricing formula

In 1996, Corrado & Su developped a methodology to correct the bias described by the volatility smile in the Black & Sholes' formula. They used a Gram-Charlier series expansion of the standard normal density function, which lead them to establish the following european call valuation formula :

$$C_{CS}(t;S) = C_{BS}(t;S) + \mu_3 Q_3 + (\mu_4 - 3)\, Q_4$$

with

$C_{BS}(t;S)$: the Black & Scholes' formula by replacing σ_{BS} with σ_{CS};

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma_{CS}^2}{2}\right)(T-t)}{\sigma_{CS}\sqrt{(T-t)}};$$

$$Q_3 = \frac{1}{6} S\, \sigma_{CS} \sqrt{(T-t)} \left[\left(2\sigma_{CS}\sqrt{(T-t)}\right) n(d_1) - \sigma_{CS}^2 (T-t)\, N(d_1)\right];$$

$$Q_4 = \frac{1}{24} S\, \sigma_{CS} \sqrt{(T-t)} \left[\begin{array}{c} \left\{d_1^2 - 1 - 3\sigma_{CS}\sqrt{(T-t)}\left(d_1 - \sigma_{CS}\sqrt{(T-t)}\right)\right\} \\ n(d_1) + \sigma_{CS}^3 (T-t)^{3/2}\, N(d_1) \end{array}\right];$$

$n(.)$: the standard normal density function;

$N(.)$: the cumulative distribution function associated to the standard normal law;

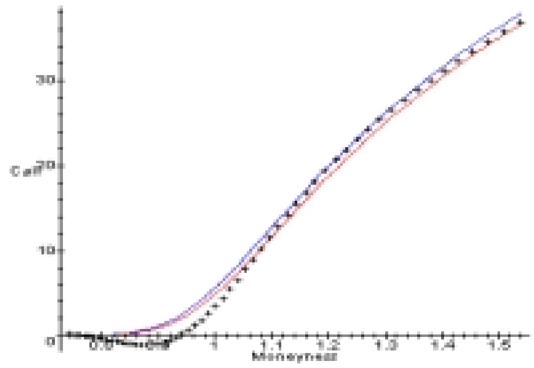
$\sigma_{CS}; \mu_3; \mu_4$: parameters of volatility, skewness and kurtosis respectively associated to the underlying's dynamic.

4.2.2 Comparison relatively to the moneyness

To compare our two factors pricing formula with that of Corrado & Su, we use the parameters' values presented in the previous section to generate call prices according to the pricing of Chauveau & Gatfaoui. Always assuming that the prices then induced are the observed market prices, we estimate the parameters σ_{CS}, μ_3 and μ_4 of the Corrado & Su's formula when considering the call prices previously computed. The estimation is achieved by numerically solving the following problem :

$$\underset{\sigma_{CS};\mu_3;\mu_4}{\mathrm{Min}} \left\{ \sum_j \left[C(t;\mathrm{moneyness}_j) - C_{CS}(t;\mathrm{moneyness}_j)\right]^2 \right\};$$

where $\mathrm{moneyness}_j = \frac{S_t}{K_j}$ with $K_j \in \{65; \ldots; 135\}$. The minimization of the sum of the squared pricing errors is carried out through a quasi-Newton method using the algorithm of Davidon-Fletcher-Powell[19], which allows to obtain the following values of parameters : $\sigma_{CS} = 0.29947828$, $\mu_3 = -1.7423441$ and $\mu_4 = 8.8830172$[20]. Introducing those estimations into the formula of Corrado & Su allows then to compute the call prices associated giving the graph underneath in which the call prices induced from the method of Chauveau & Gatfaoui, the formula of Corrado & Su and the valuation of Black & Scholes are respectively drawn in blue, black and red.



European call pricing according to the three formulae presented.

Considering the graph and omitting the negative pricing problem presented by the formula of Corrado & Su for small values of the moneyness, we observe that the formula of Corrado & Su seems to lie between our two factors valuation formula and the Black & Scholes' formula.

5 Conclusion

In this paper, we have proposed an analytical formula for valuing a european call with two risk factors : the first factor corresponds to the systematic risk and the second factor corresponds to the specific risk of the underlying asset, as in Sharpe's simplified model[21]. We derived our valuation formula from a risk disaggregation in the Black & Scholes' pricing formula, which allowed us to get an analytical expression for the european call pricing. The parameters

[19] For further explanation, the reader is invited to consult the book of Press, Flannery, Teukolsky & Vetterling (1989).

[20] Note that those values show that, on one hand, the distribution of the call prices is left-skewed, and on the other hand, this distribution is described by an excess of kurtosis relatively to a normal law, which indicates the existence of a left fat tail in the distribution.

[21] This decomposition could also be improved if we consider that systematic risk and specific risk are themselves some aggregation of respectively two different series of influent variables (i. e. : multifactor framework)

of this formula therefore depend on the volatilities of the two risk factors, or, alternatively, on the volatility of the market factor and on that of the stock.

We then built a market factor replicating portfolio which is a naively divesi…ed portfolio and we studied the modi…cations induced in the CAPM framework. We found that, under some regularity conditions, the diversi…cation e¤ect known to o¤set the speci…c risk applies. The price of a european call on a stock may then be expressed in terms of the volatilities of the MFR portfolio and of the underlying stock (and of its beta).

Finally, we made a few simulations in order to compare our analytical formula with those of Black & Scholes and of Corrado & Su. First, the comparison with the formula of Black & Scholes underlines the fact that the distinction between the systematic risk and the idiosyncratic risk brings an additional degree of accuracy in the european call valuation. Moreover, assuming that the european call prices induced from our two factors formula are correct and obtaining the implicit volatility by inverting the Black & Scholes' formula, leads to the evidence of a volatility smile. Second, the comparison with the valuation method proposed by Corrado & Su shows that european call prices induced from the pricing of Chauveau & Gatfaoui exhibit skewness and kurtosis characteristics in accordance with the observed market behavior. Furthermore, the results generated by simulations seem to suggest that the formula of Corrado & Su lies between our two factors valuation formula and the pricing formula of Black & Scholes.

The results in this paper are to be completed by a test on empirical data. This is all the more essential that we have imposed a volatility constraint when comparing our formula with that of Black & Scholes.

6 Appendix : The call pricing with two risk factors

We introduce here the pricing framework of the call, which is the following one : Analogous to the no arbitrage opportunity principle, we have the next relation under the risk neutral measure Q associated to the valuation of the bidimensional process $Z_t = \begin{pmatrix} X_t \\ I_t \end{pmatrix}'$:

$\forall t \in [0;T]$;

$$\begin{aligned} C(t;X_t;I_t) &= C(t;Z_t) = E_Q\left[e^{-r(T-t)}C(T;Z_T)\,|\,F_t\right] \\ &= e^{-r(T-t)}E_Q\left[C(T;Z_T)\,|\,Z_t\right] \end{aligned}$$

which can be written :

$$C(t;Z) = e^{-r(T-t)} \int_{\mathbb{R}\times\mathbb{R}} \left\{ \left(\max\left\{ a'\cdot \underline{e}^{Y_T}, b'\cdot \underline{e}^{Y_T} \right\} - K \right)^+ f(Y_T | Y_t) \right\} dY_{T;1}dY_{T;2} \quad (9)$$

where

$$a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \forall W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \in \mathbb{R}^2; \underline{e}^{T} = \begin{pmatrix} e^{W_1} \\ e^{W_2} \end{pmatrix}$$

and $f(Y_T|Y_t)$ represents the conditional density of Y_T given Y_t. We know that the law of Y_T given Y_t corresponds to a bidimensional normal law $N(MOY_t; VAR_t)$ in a risk neutral universe such that :

$$MOY_t = Y_t + (T-t)H_t = \begin{pmatrix} Y_t^1 \\ Y_t^2 \end{pmatrix} + (T-t) \begin{bmatrix} r - \frac{\sigma^2}{2} \\ r - \frac{\sigma^{*2}}{2} \end{bmatrix}$$

$$\Sigma_t = \Sigma(t;X_t;I_t) = \begin{pmatrix} X_t\sigma(t;X_t) & 0 \\ 0 & I_t\sigma^*(t;I_t) \end{pmatrix}$$

$$VAR_t = \Sigma(t;X_t;I_t)\Sigma'(t;X_t;I_t)(T-t) = (T-t)\Sigma_t\Sigma_t'$$

Knowing the law of Y_T given Y_t, we can compute the integral of the relation (9) after introducing the following notations :

$$\begin{aligned} g(Z_T) &= g(\underline{e}^{Y_T}) = \left(\max\left\{ a'\cdot \underline{e}^{Y_T}, b'\cdot \underline{e}^{Y_T} \right\} - K \right)^+ \\ &= 1_{\mathcal{D}}\, h(\underline{e}^{Y_T}) \end{aligned}$$

with

$$\begin{aligned} 1_{\mathcal{D}} &= \text{indicator function of the set } \mathcal{D} \\ \mathcal{D} &= \left\{ \left[\max\left\{ a'\cdot \underline{e}^{Y_T}, b'\cdot \underline{e}^{Y_T} \right\} - K \right] \geq 0 \right\} \\ h(\underline{e}^{Y_T}) &= \left(\max\left\{ a'\cdot \underline{e}^{Y_T}, b'\cdot \underline{e}^{Y_T} \right\} - K \right) \end{aligned}$$

The relation (6) is written :

$$C(t;X_t;I_t) = e^{-r(T-t)} \int_{\mathbb{R}^2} 1_{\mathcal{D}} \left[h(\underline{e}^{Y_T}) f(Y_T|Y_t) \right] dY_{T;1}dY_{T;2} \quad (10)$$

Or

$$C(t;X_t;I_t) = e^{-r(T-t)} \int_{=} \left\{ h(e^{Y_T}) f(Y_T | Y_t) \right\} dY_{T;1}dY_{T;2} \tag{11}$$

We now explain how to calculate the call price with two risk factors when achieving the following integration[22] :

$$\mathcal{C} = \frac{C(t;Z)}{e^{-r(T-t)}} = \int_{u \in \mathbb{R}^2; =} \left(\begin{array}{c} h(e^{u+MOY}) \frac{1}{2\pi(T-t)\sigma\sigma\alpha} \\ \exp\left[-\frac{1}{2(T-t)} \left(\frac{u_1^2}{\sigma^2} + \frac{u_2^2}{\sigma\alpha 2} \right) \right] \end{array} \right) du_1 du_2$$

$$\mathcal{C} = \int_{u \in \mathbb{R}^2; =} \left\{ \begin{array}{c} \left(\alpha \left(a' \cdot e^{u+MOY} \right) - \left(b' \cdot e^{u+MOY} \right) \right) - K \\ \frac{1}{2\pi(T-t)\sigma\sigma\alpha} \exp\left[-\frac{1}{2(T-t)} \left(\frac{u_1^2}{\sigma^2} + \frac{u_2^2}{\sigma\alpha 2} \right) \right] \end{array} \right\} du_1 du_2$$

$$\mathcal{C} = \int_{u \in \mathbb{R}^2; =} \left\{ \begin{array}{c} \left(\alpha \left(a' \cdot e^{u+MOY} \right) - \left(b' \cdot e^{u+MOY} \right) \right) \\ \frac{1}{2\pi(T-t)\sigma\sigma\alpha} \exp\left[-\frac{1}{2(T-t)} \left(\frac{u_1^2}{\sigma^2} + \frac{u_2^2}{\sigma\alpha 2} \right) \right] \end{array} \right\} du_1 du_2$$

$$- K \int_{u \in \mathbb{R}^2; =} \left\{ \frac{1}{2\pi(T-t)\sigma\sigma\alpha} \exp\left[-\frac{1}{2(T-t)} \left(\frac{u_1^2}{\sigma^2} + \frac{u_2^2}{\sigma\alpha 2} \right) \right] \right\} du_1 du_2$$

We pose $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$ with :

$$\mathcal{C}_1 = \int_{u \in \mathbb{R}^2; =} \left\{ \begin{array}{c} \left(\alpha \left(a' \cdot e^{u+MOY} \right) - \left(b' \cdot e^{u+MOY} \right) \right) \\ \frac{1}{2\pi(T-t)\sigma\sigma\alpha} \exp\left[-\frac{1}{2(T-t)} \left(\frac{u_1^2}{\sigma^2} + \frac{u_2^2}{\sigma\alpha 2} \right) \right] \end{array} \right\} du_1 du_2$$

$$\mathcal{C}_2 = - K \int_{u \in \mathbb{R}^2; =} \left\{ \frac{1}{2\pi(T-t)\sigma\sigma\alpha} \exp\left[-\frac{1}{2(T-t)} \left(\frac{u_1^2}{\sigma^2} + \frac{u_2^2}{\sigma\alpha 2} \right) \right] \right\} du_1 du_2$$

$$U_T | U \sim N(0; VAR)$$

$$\begin{aligned} = &= \left\{ \left[\alpha \left(a' \cdot e^{u_1 + MOY_1} \right) - \left(b' \cdot e^{u_2 + MOY_2} \right) \right] - K \geq 0 \right\} \\ &= \left\{ \beta (u_1 + MOY_1) + (u_2 + MOY_2) \geq \ln \left(\frac{K}{\alpha} \right) \right\} \end{aligned}$$

[22] To simplify the proof, we forget the time indexation at the current date.

First member integration :

$$\Phi_1 = \alpha \int_{u\in\mathbb{R}^2;=} \left\{ \frac{e^{-\beta u_1} e^{u_2}}{2\pi (T-t)\sigma\sigma^*} \exp\left[-\frac{1}{2(T-t)} \left(\frac{u_1^2}{\sigma^2} + \frac{u_2^2}{\sigma^{*2}} \right) \right] \right\} du_1 du_2$$

with

$$\alpha = \Lambda X^{\beta} I e^{\left[-\beta\left(r-\frac{\sigma^2}{2}\right) + \left(r-\frac{\sigma^{*2}}{2}\right) \right](T-t)}$$

Applying the Fubini theorem, we can write the following relation about the first member :

$$\Phi_1 = \alpha \int_{\mathbb{R}} \frac{e^{-\beta u_1} e^{-\frac{1}{2(T-t)}\frac{u_1^2}{\sigma^2}}}{\sqrt{2\pi (T-t)}\sigma} J_1(u_1) du_1$$

with

$$J_1(u_1) = \int_d^{+\infty} \frac{e^{u_2}}{\sqrt{2\pi (T-t)}\sigma^*} e^{-\frac{1}{2(T-t)}\frac{u_2^2}{\sigma^{*2}}} du_2$$

$$d = \ln\left(\frac{K}{\Lambda X^{\beta} I} \right) - \beta u_1 - \left[\beta\left(r-\frac{\sigma^2}{2}\right) + \left(r-\frac{\sigma^{*2}}{2}\right) \right](T-t)$$

An appropriate change of variable allows us to establish the following result :

$$J_1(u_1) = e^{\frac{1}{2}\sigma^{*2}(T-t)} N\left(\bar{d}\right)$$

with

$$\bar{d} = \bar{d}(u_1) = \frac{\ln\left(\frac{\Lambda X^{\beta} I}{K}\right) + \beta u_1 + \left[\beta\left(r-\frac{\sigma^2}{2}\right) + \left(r+\frac{\sigma^{*2}}{2}\right) \right](T-t)}{\sigma^* \sqrt{(T-t)}}$$

and $N(.)$ the standard normal cumulative distribution function.

A second change of variable under the integral concerning u_1 allows then to pose :

$$\Phi_1 = \gamma \int_{\mathbb{R}} \frac{e^{-\frac{1}{2}w^2}}{\sqrt{2\pi}} N\left(\bar{d}_1\right) dw$$

with

$$\hat{d}_1 = \hat{d}_1(w)$$

$$= \frac{\ln\left(\frac{\Lambda X^{\beta} l}{K}\right) + w\beta\sigma\sqrt{(T-t)} + \left[\beta\left(r - \frac{\sigma^2}{2}\right) + \beta^2\sigma^2 + \left(r + \frac{\sigma_\Lambda^2}{2}\right)\right](T-t)}{\sigma_\Lambda\sqrt{(T-t)}}$$

$$\gamma = \Lambda X^{\beta} l\, e^{\left[\beta\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{2}\beta^2\sigma^2 + r\right](T-t)} = S e^{\left[\beta\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{2}\beta^2\sigma^2 + r\right](T-t)}$$

Second member integration :

$$\hat{C}_2 = -K \int_{u \in \mathbb{R}^2;} \left\{ \frac{1}{2\pi(T-t)\sigma\sigma_\Lambda} \exp\left(-\frac{1}{2(T-t)} \left(\frac{u_1^2}{\sigma^2} + \frac{u_2^2}{\sigma_\Lambda^2} \right) \right) \right\} du_1 du_2$$

$$= -K \int_{\mathbb{R}} \frac{e^{-\frac{1}{2(T-t)}\frac{u_1^2}{\sigma^2}}}{\sigma\sqrt{2\pi(T-t)}} J_2(u_1) du_1$$

$$J_2(u_1) = \int_{d}^{+\infty} \frac{e^{-\frac{1}{2(T-t)}\frac{u_2^2}{\sigma_\Lambda^2}}}{\sigma_\Lambda\sqrt{2\pi(T-t)}} du_2$$

Applying an appropriate change of variable (to get the standard normal distribution density), we obtain :

$$J_2(u_1) = N\left(\hat{d}_2\right)$$

with

$$\hat{d}_2 = \hat{d}_2(u_1) = \frac{\ln\left(\frac{\Lambda X^{\beta} l}{K}\right) + \beta u_1 + \left[\beta\left(r - \frac{\sigma^2}{2}\right) + \left(r - \frac{\sigma_\Lambda^2}{2}\right)\right](T-t)}{\sigma_\Lambda\sqrt{(T-t)}}$$

This gives us the following relation :

$$\hat{C}_2 = -K \int_{\mathbb{R}} \frac{e^{-\frac{1}{2(T-t)}\frac{w^2}{\sigma^2}}}{\sigma\sqrt{2\pi(T-t)}} N\left(\hat{d}_2\right) dw$$

$$\hat{d}_2 = \hat{d}_2(w)$$

Consequently, the price of the considered european call has the following form :

$$\hat{C} = \gamma \int_R \frac{e^{-\frac{1}{2}w^2}}{\sqrt{2\pi}} N\left(\hat{d}_1\right) dw - K \int_R \frac{e^{-\frac{1}{2(T-t)}\frac{w^2}{\sigma^2}}}{\sigma\sqrt{2\pi(T-t)}} N\left(\hat{d}_2\right) dw$$

with

$$\gamma = X^{*\beta} I e^{\left[\beta\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{2}\beta^2\sigma^2+r\right](T-t)} = S e^{\left[\beta\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{2}\beta^2\sigma^2+r\right](T-t)}$$

$$\hat{d}_1 = \hat{d}_1(w) = \frac{\ln\left(\frac{X^{*\beta} I}{K}\right) + w\beta\sigma\sqrt{(T-t)} + \left[\beta\left(r-\frac{\sigma^2}{2}\right)+\beta^2\sigma^2+\left(r+\frac{\sigma^{*2}}{2}\right)\right](T-t)}{\sigma^*\sqrt{(T-t)}}$$

$$\hat{d}_2 = \hat{d}_2(w) = \frac{\ln\left(\frac{X^{*\beta} I}{K}\right) + \beta w + \left[\beta\left(r-\frac{\sigma^2}{2}\right)+\left(r-\frac{\sigma^{*2}}{2}\right)\right](T-t)}{\sigma^*\sqrt{(T-t)}}$$

with

$$\hat{d}_1 = \hat{d}_2 + \frac{\beta w\left(\sigma\sqrt{(T-t)}-1\right)+\left[\beta^2\sigma^2+\sigma^{*2}\right](T-t)}{\sigma^*\sqrt{(T-t)}}$$

REFERENCES

Black F., 1972, Capital Market Equilibrium with Restricted Borrowing, Journal of Business,July, p. 444 -455.

Black F. & M. Scholes, 1973, The Pricing of Options and Corporate Liabilities, Journal of Political Economy, p. 637 - 653.

Cochrane J. H. & J. Saa-Requejo, 1998, Beyond Arbitrage : " Good Deal " Asset Price Bounds in Incomplete Markets, Working Paper, University of Chicago.

Corrado C. J. & T. Su, 1997, Implied Volatility Skews and Stock Return Skewness and Kurtosis Implied by Stock Option Prices, European Journal of Finance, vol. 3, p. 73 - 85.

Corrado C. J. & T. Su, 1996, Skewness and Kurtosis in S&P 500 Index Returns Implied by Option Prices, Journal of Financial Research, vol. 19, p. 175 - 192.

Cox J. C., Ross S. A. & M. Rubinstein, 1979, Option Pricing : A Simplified Approach, Journal of Financial Economics, vol. 7.

Daves P. R., Ehrhardt M. C. & R. A. Kankel, 2000, Estimating Systematic Risk : The Choice of Return Interval and Estimation Period, Journal of Financial and Strategic Decisions, vol. 13, p. 15 - 34.

Heston S., 1993, A Closed Form Solution for Options with Stochastic Volatility with Applications to Bonds and Currency Options, Review of Financial Studies, 6, p. 327 - 344.

Lintner J., 1965, Valuation of Risk Assets and The Selection of Risky Investments in Stock Portfolios and Capital Budgets, Review of Economics and Statistics, p. 13 - 37.

Lintner J., 1969, The Aggregation of Investor's Diverse Judgments and Preferences in Purely Competitive Security Markets, Journal of Financial and Quantitative Analysis, December, p. 347 - 400.

Lucas A., Klaassen P., Spreij P. & S. Straetmans, 2001, Tail Behavior of Credit Loss Distributions for General Latent Factor Models, Tinbergen Institute Working Paper.

Malkiel B. G. & Y. Xu, 2001, Idiosyncratic Risk and Security Returns, Department of Economics, Princeton University Working Paper.

Merton R. C., 1973, An Intertemporal Capital Asset Pricing Model, Econometrica, vol. 41, p. 867 - 887.

Mossin J., 1966, Equilibrium in a Capitral Asset Market, Econometrica, October, p.768 - 783.

Pham H., 1998, Méthodes d'Evaluation et Couverture en Marché Incomplet, CREST - ENSAE.

Press W. H., Flannery B. P., Teukolsky S. A. & W. T. Vetterling, 1989, Numerical Receipes in Pascal, Cambridge University Press.

Quittard-Pinon F., 1993, Marchés des Capitaux et Théorie Financière, Economica.

Rogers L. C. G. & D. Williams, 1994, Diffusions, Markov Processes and Martingales : Itô Calculus, Cambridge University Press (Volume 2).

Rogers L. C. G. & D. Williams, 1994, Diffusions, Markov Processes and Martingales : Foundations, Cambridge University Press (Volume 1).

Sharpe W. F., 1970, Portfolio Theory and Capital Markets, Mc Graw-Hill, New-York.

Sharpe W. F., 1963, A Simpli...ed Model For Portfolio Analysis, Management Science, vol. 9.

Sharpe W. F., 1964, Capital Asset Prices : A Theory of Market Equilibrium Under Conditions of Risk, Journal of Finance, vol. 19, p. 425 - 442.

Treynor., 1961, Toward a Theory of the Market Valuue of Risky Assets, Unpublished manuscript.

Wilson T. C., 1998, Portfolio Credit Risk, FRBNY Economic Review, p. 71 - 82.

*
* *