

Valuation of Equity-Linked Life Insurance Contracts with Flexible Guarantees in a Non Gaussian Economy

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Abstract

In this paper, we focus on the pricing of a particular life insurance contract where the conditional payoff to the policyholder is the maximum of two risky assets. The first one has larger expected returns but is riskier while the second one is less risky but still can earn more than an investment in a risk-free asset. Of course this payoff can be seen as the result of an investment in the first asset and a long position in an exchange option. The latter was priced under Gaussian assumptions by Margrabe (1978). To take kurtosis into account the underlying dynamics have to be changed. In this paper, we suggest modelling the underlying dynamics of the second asset by a simple diffusion, i.e. a geometric Brownian motion with a low volatility while the riskier asset follows a jump diffusion. More precisely, this process has a Brownian component and a compound Poisson one, where jump size is driven by a double exponential distribution. This stochastic process introduced by Kou (2002) is easy to manage and proves to be a versatile tool. To price our life insurance contract, we use a generalized Fourier transform and obtain the solution numerically. As far as we know, this is the first paper to use this approach. This methodology proves to be very efficient both with respect to accuracy and to computational time. We also consider a contract with a fixed guarantee and price it while taking into account stochastic volatility and jumps. We incorporate mortality using a classical Makeham law.

Keywords: Life insurance contracts, Mortality models, Stochastic volatility, Jumps.

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1 Introduction

Known under different names such as variable annuities in the USA, segregated funds in Canada or unit linked insurance in the UK, numerous life insurance contracts have been marketed by insurance companies with great success. They represent a multi billion US dollar market. These contracts combine both financial and actuarial features. They are linked to financial markets and to policy-holder's life. Their payoffs generally are associated with the performance of a financial index or a reference portfolio. Pricing and hedging these contracts is not an easy task, for many reasons and in particular because they involve embedded options. So many papers are devoted to their analysis. Research in this area is abundant and began with Brennan and Schwartz (1976) and never ends since then. Important papers in this area are those of Briys and de Varenne (1997), Bacinello and Ortu (1993), Tankanen and Lukkarinen (2003), Bacinello (2005), Bernard, Le Courtois, and Quittard-Pinon (2005), and Ballotta (2005). A large part of these articles are conducted under the assumption of Gaussian returns. However now it is well recognized that financial asset returns generally cannot be represented by normal laws, because of empirical leptokurtic distribution. Non-normal and in particular Lévy processes have recently been introduced by Kassberger, Kiesel, and Liebmann (2007), and by Le Courtois and Quittard-Pinon (2008). In this paper we focus on the pricing of a particular life insurance contract where the payoff due to the policyholder in case of life is the maximum of two risky assets the first one is considered to have larger expected returns but is riskier and the second one can be adopted because it is less risky but can

earn more than an investment in a risk-free asset. Of course this payoff can be seen as the result of an investment in the first asset and a long position in an exchange option. This one has been priced under Gaussian assumptions by Margrabe (1978). To take into account kurtosis the underlying dynamics have to be changed. To take into account this fact, in a recent paper, Minjing Li proposes to use a Gram-Charlier development. In this paper and in the actuarial context we consider, we suggest to model the underlying dynamics of the second asset by a simple diffusion, i.e. a geometric Brownian motion with a low volatility while the riskier asset follows a jump diffusion. More precisely, this process has a Brownian component and a compound Poisson one, where jump size is driven by a double exponential distribution. This stochastic process introduced by Kou (2002) is easy to manage and proves to be a versatile tool. To price our life insurance contracts, we use a generalized Fourier transform approach and obtain numerically the solution. As far as we know, it is the first paper to use this approach. This methodology proves to be very efficient both with respect to accuracy and to computational time. We also consider an alternative approach using stochastic for contracts with simple guarantee. We incorporate mortality using a Makeham law. This paper is organized as follows. In a first section we recall the mortality model used throughout the article. In section 2 we analyse the contract with a simple guarantee, we call it a Guarantee Minimum Maturity Benefit, GMMB, and we price it under different modeling for the evolution of the financial market, whilst section 3 is devoted to the contract with a flexible guarantee. A conclusion ends the paper.

2 Mortality risk

We present in this section the model we use to take into account the mortality risk.

2.1 Mortality model

Hereafter, we adopt usual actuarial notations. The future lifetime of a policyholder aged x is the random variable T_x . For an individual aged x , the probability of death before time $t \geq 0$ is $P(T_x \leq t) = 1 - (t p_x)$. Let λ denote the force of mortality, we have

$$P(T_x \leq t) = 1 - \exp\left(-\int_0^t \lambda(x+s)ds\right). \quad (1)$$

As usual, $F_x(t)$ and $f_x(t)$ are respectively the c.d.f. and the p.d.f. of the random variable T_x . We recall the well-known relationship:

$$\lambda(x+t) = \frac{f_x(t)}{1 - F_x(t)}, \quad (2)$$

see for example Gerber (1997) or Bowers, Gerber, Hickman, Jones, and Nesbitt (1997).

We chose a mortality model from the Gompertz-Makeham distribution family. In particular, the parametrized Makeham mortality model has the following force of mortality at age x :

$$\lambda(x) = A + B.C^x, \quad (3)$$

where $B > 0, C > 1$ and $A \geq -B$.

Table 1 – Makeham mortality model parameters for the US – Melnikov and Romaniuk (2006)

A	B	C
9.566×10^{-4}	5.162×10^{-5}	1.09369

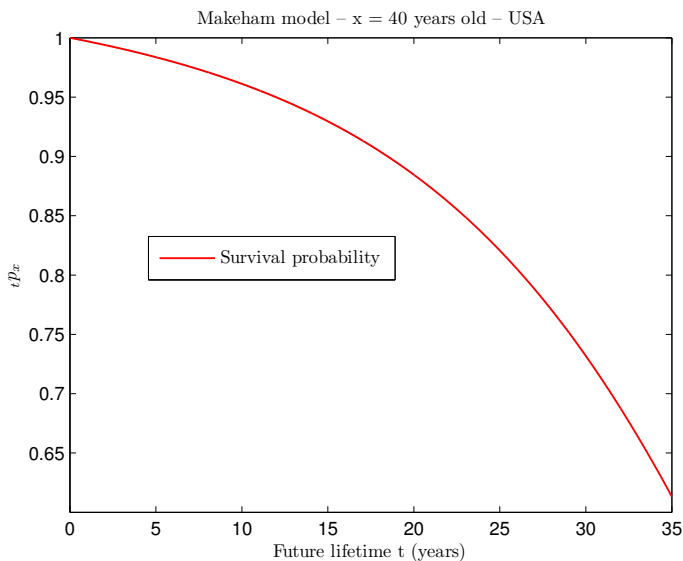


Figure 1 – Survival probability of a 40 year old individual, according to a Makeham mortality model, as calibrated by Melnikov and Romaniuk (2006).

To ease notation, we generally omit the x from the future lifetime r.v. and only write T when no confusion is possible. We assume stochastic independence between mortality and financial risks.

2.2 Distribution parameters and survival probabilities

We use the distribution parameters obtained by Melnikov and Romaniuk (2006) from the Human mortality database¹ 1959-1999 mortality data, with no distinction made between females and males. The estimated Makeham parameters for the USA are as given in Table 1.

¹The Human mortality database is available at <http://www.mortality.org>

We plot in Fig. 1 the survival probability ${}_t p_x$ of an individual aged $x = 40$ for a given residual lifetime of t years, according to the Makeham mortality model given previously.

3 Guaranteed Minimum Maturity Benefit (GMMB)

The Guaranteed Minimum Maturity Benefit (or GMMB) is an equity-linked life insurance contract whereby the insurer pays upon expiry the current value of the underlying account if the policyholder survives the term with the benefit of a guaranteed return rate on the notional amount, should the market turn downwards.

3.1 GMMB modelling and pricing

Let us denote by S the price process of the underlying asset and by d its continuous dividend yield. The notional amount of the guarantee is given by S_0 . The time to maturity is denoted by Θ : it is the remaining time up to expiry date. Thus a policyholder aged x when buying a contract will be $x + \Theta$ year old at expiry.

We also assume a constant continuously compounded risk-free interest rate r . The guaranteed rate of return is g . It can also be seen as a fraction φ of the risk-free interest rate. The payoff at expiry writes

$$\max\{S_\Theta, S_0 e^{g\Theta}\} \mathbf{1}_{\{T_x > \Theta\}} = \left(L + [S_\Theta - L]^+ \right) \mathbf{1}_{\{T_x > \Theta\}}, \quad (4)$$

where $L = S_0 e^{g\Theta} = S_0 e^{\varphi r \Theta}$ denotes the level of guarantee at expiry.

This payoff is composed of a contractual guarantee L and a long position in a European call option with strike L , conditional on the policyholder being

alive at the contract expiry date.

Based on our hypothesis of independence between mortality risk and financial risk, it remains to price the corresponding European call option. With our subsequent choice of models for the underlying asset, the market is not necessary complete. We consider a suitable risk-neutral measure has been chosen and we use it as a pricing measure. A usual way to choose such a measure is to use the measure implied by the market as explained by Björk (2004). The corresponding risk-neutral universe is given by the stochastic basis $(\Omega, \mathcal{A}, \mathcal{F}_{t,t \geq 0}, Q)$, where (Ω, \mathcal{A}, Q) is a probability space with the risk-neutral measure Q , and $(\mathcal{F}_t, t \geq 0)$ is a filtration generated by the asset price process S .

The fair value of the GMMB contract at inception is thus given by

$$\begin{aligned} GMMB &= E_Q \left[e^{-r\Theta} \left(L + [S_\Theta - L]^+ \right) \mathbf{1}_{\{T_x > \Theta\}} \right] \\ GMMB &= {}_{\Theta}p_x E_Q \left[e^{-r\Theta} \left(L + [S_\Theta - L]^+ \right) \right]. \end{aligned}$$

As L is a fixed guarantee level, the leftmost part of the expectation becomes

$$E_Q \left[e^{-r\Theta} L \right] = L e^{-r\Theta}, \quad (5)$$

while the remaining part is the present value of a European call option with strike L . Eventually, we get the value of the GMMB contract as:

$$GMMB = {}_{\Theta}p_x \left(L e^{-r\Theta} + C(S_0, L) \right), \quad (6)$$

where $C(S_0, L)$ is the value of a European call option with strike L .

In all the following underlying asset models, the price process S can be expressed as $S_t = S_0 \exp(X_t)$ and the characteristic function of the process X can be obtained each time as:

$$\phi_{X_t}(u) = E_Q \left[e^{iuX_t} \right] = e^{\psi_{X_t}(u)} \quad (7)$$

where ψ_{X_t} is called the characteristic exponent of X . In this case, the European call option can be priced using a generalized Fourier approach as in Quittard-Pinon and Randrianarivony (2008b).

To do so, we need first to ensure that the discounted forward price process itself is a martingale under the risk-neutral measure Q . Indeed, we have:

$$\begin{aligned} S_0 &= E_Q \left[e^{-rt} (S_t e^{dt}) \right] \\ &= E_Q \left[e^{-(r-d)t} S_0 e^{X_t} \right] \\ S_0 &= S_0 e^{-(r-d)t} e^{\psi_{X_t}(-i)}. \end{aligned}$$

Thus the characteristic exponent of X must obey the following equation:

$$-(r-d)t + \psi_{X_t}(-i) = 0. \quad (8)$$

We call Eq. (8) the equivalent martingale measure condition.

After ensuring that the model parameters obey the equivalent martingale measure condition, we can obtain the European call option price in Eq. (6)

with the following formula:

$$C(S_0, L) = L \frac{1}{2\pi} \int_{\mathbb{R}+i\delta} \frac{e^{ixu-r\Theta+\psi_{X_\Theta}(u)}}{(-iu)(-iu+1)} du, \quad (9)$$

where $x = \ln(S_0/L)$ is the log-moneyness and $\delta < -1$.

3.2 GMMB value under various underlying models

We review various models for the underlying asset price in this section. For illustration purposes, we consider an insured aged $x = 40$. The interest rate is at $r = 0.05$. The underlying asset delivers a continuous dividend yield of $d = 0.01$ and its initial value is set at $S_0 = 1$.

3.2.1 Geometric Brownian motion dynamics

We consider the classical case where the underlying asset price process is a Geometric Brownian motion. It is a pure diffusion stochastic process with volatility σ . The corresponding characteristic exponent has the following form:

$$\psi_{X_t}(u) = t \left(i\mu u - \frac{\sigma^2}{2} u^2 \right), \quad (10)$$

where μ is a drift parameter.

To ensure that the equivalent martingale measure condition (8) holds, we set the μ parameter as follows:

$$\mu = r - d - \frac{\sigma^2}{2}.$$

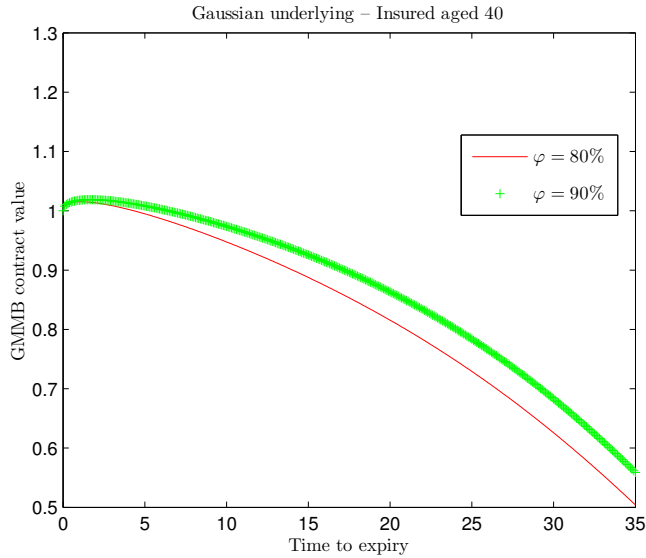


Figure 2 – GMMB value with respect to time to expiry Θ . Gaussian underlying with volatility $\sigma = 0.071$.

3.2.2 Kou jump diffusion model

Instead of the previous purely diffusive process, we take into account price discontinuities by adding jumps to the model. The resulting model is a jump diffusion model where jumps occur as Poisson events at a rate of λ . In this article, we choose jumps with a double exponential distribution as proposed by Kou (2002). More precisely, jumps are supposed to be i.i.d. with the following double exponential law:

$$f_J(y) = p\lambda_1 e^{-\lambda_1 y} 1_{y>0} + q\lambda_2 e^{\lambda_2 y} 1_{y\leq 0} \quad (11)$$

with $p \geq 0$, $q \geq 0$, $p + q = 1$, $\lambda_1 > 0$ and $\lambda_2 > 0$.

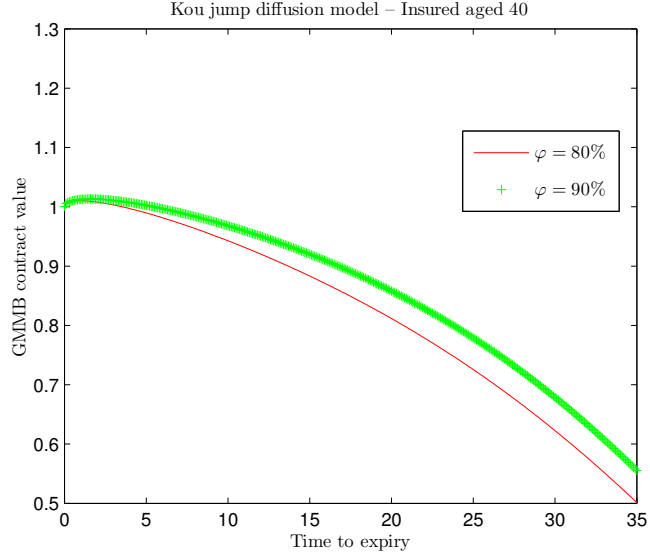


Figure 3 – GMMB value with respect to time to expiry Θ . Kou jump diffusion model with parameters $\sigma = 0.044$, $\lambda = 0.2$, $p = 0.4$, $\lambda_1 = 15$, and $\lambda_2 = 10$.

The Kou jump diffusion model characteristic exponent sums up to:

$$\psi_{X_t}(u) = t \left(i\mu u - \frac{\sigma^2}{2} u^2 + \lambda \left(\frac{p\lambda_1}{\lambda_1 - iu} + \frac{q\lambda_2}{\lambda_2 + iu} - 1 \right) \right). \quad (12)$$

The equivalent martingale measure condition (8) entails:

$$\mu = r - d - \frac{\sigma^2}{2} - \lambda \left(\frac{p\lambda_1}{\lambda_1 - 1} + \frac{q\lambda_2}{\lambda_2 + 1} - 1 \right).$$

3.2.3 Stochastic volatility model

In this case, we use the Heston (1993) model as described by the following stochastic differential equations in the risk-neutral universe:

$$dS_t = (r - d)S_t dt + \sqrt{v_t} S_t dZ_t \quad (13)$$

and

$$dv_t = -\kappa(v_t - \bar{v})dt + \eta\sqrt{v_t}dW_t \quad (14)$$

where v is the variance process of the underlying S , η is the volatility of variance, and ρ is the correlation between the two Brownian motions Z and W . The speed of reversion to the long-term mean variance \bar{v} is κ .

Define the following functions of u :

$$\begin{aligned} \alpha &= -\frac{u^2}{2} - \frac{iu}{2} \\ \beta &= \kappa - \rho\eta iu \\ \gamma &= \frac{\eta^2}{2} \\ \Delta &= \sqrt{\beta^2 - 4\alpha\gamma} \end{aligned}$$

and

$$\begin{aligned} r_{\pm} &= \frac{\beta \pm \Delta}{2\gamma} \\ g &= \frac{r_-}{r_+}. \end{aligned}$$

Define also

$$C(u, t) = \kappa \left[r_- t - \frac{1}{\gamma} \ln \left(\frac{1 - g e^{-\Delta t}}{1 - g} \right) \right] \quad (15)$$

and

$$D(u, t) = r_- \frac{1 - e^{-\Delta t}}{1 - g e^{-\Delta t}}, \quad (16)$$

the characteristic exponent in the case of the stochastic volatility model is

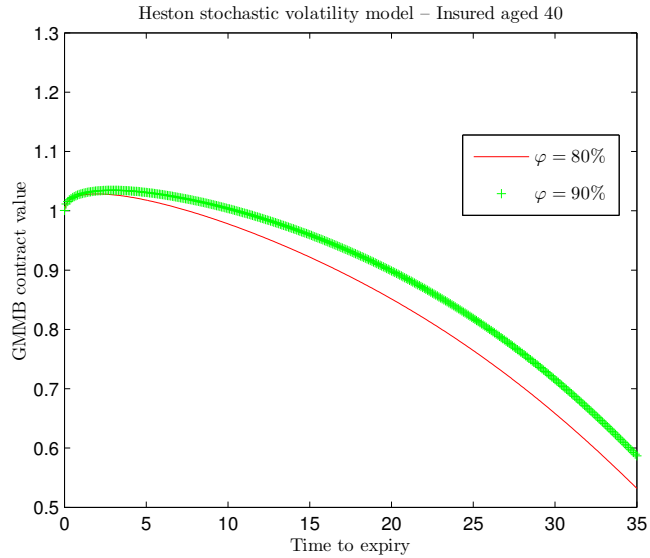


Figure 4 – GMMB value with respect to time to expiry Θ . Heston stochastic volatility model with parameters $v_0 = 0.01, \bar{v} = 0.01, \kappa = 2, \eta = 0.1, \rho = -0.5$.

given by

$$\psi_{X_t}(u) = i\mu u t + C(u, t)\bar{v} + D(u, t)v_0, \quad (17)$$

where v_0 is the spot variance.

As the value of the function α at the complex value $-i$ is zero, so are the values of r_- and g . Thus, the equivalent martingale measure condition (8) is satisfied if

$$\mu = r - d.$$

3.2.4 Stochastic volatility model with Kou jumps

In this setting, we take into account both stochastic volatility and jump effects. As noted in Bakshi, Cao, and Chen (1997), they capture most stylized facts for the underlying model.

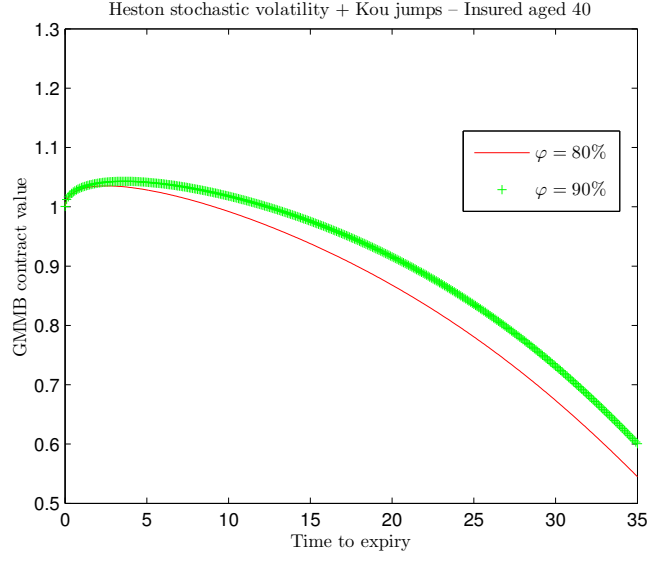


Figure 5 – GMMB value with respect to time to expiry Θ . Insured aged 40. Heston stochastic volatility with Kou jumps. Parameters $v_0 = 0.01$, $\bar{v} = 0.01$, $\kappa = 2$, $\eta = 0.1$, $\rho = -0.5$, $\lambda = 0.2$, $p = 0.4$, $\lambda_1 = 15$, and $\lambda_2 = 10$.

The characteristic exponent is given by:

$$\psi_{X_t}(u) = i\mu u t + C(u, t)\bar{v} + D(u, t)v_0 + \lambda t \left(\frac{p\lambda_1}{\lambda_1 - iu} + \frac{q\lambda_2}{\lambda_2 + iu} - 1 \right). \quad (18)$$

As for the equivalent martingale measure condition (8), we have:

$$\mu = r - d - \frac{\sigma^2}{2} - \lambda \left(\frac{p\lambda_1}{\lambda_1 - 1} + \frac{q\lambda_2}{\lambda_2 + 1} - 1 \right).$$

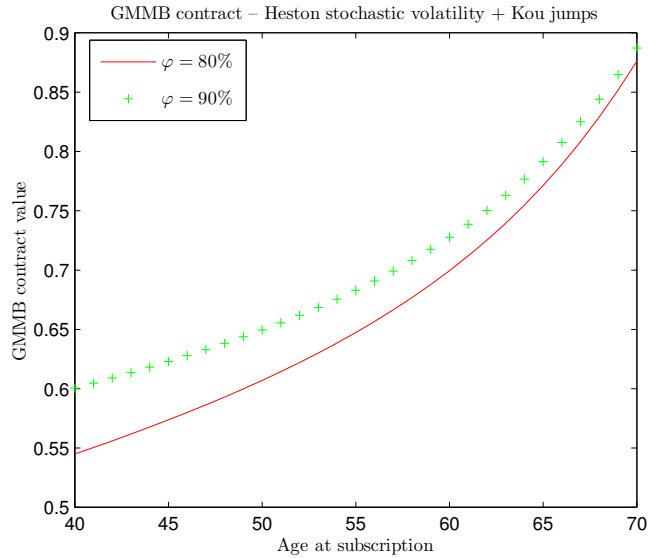


Figure 6 – GMMB value with respect to age at subscription. Heston stochastic volatility with Kou jumps. Parameters $v_0 = 0.01$, $\bar{v} = 0.01$, $\kappa = 2$, $\eta = 0.1$, $\rho = -0.5$, $\lambda = 0.2$, $p = 0.4$, $\lambda_1 = 15$, and $\lambda_2 = 10$.

4 Flexible guarantee contract

The contract we consider here is a Pure Endowment with a Flexible guarantee policy. It gives in case of life a variable capital to the policy-holder. In fact contrarily to the fixed guarantee contract where the amount given is perfectly known at the contract inception, in the PEFG the insurer gives at the contract expiry the maximum between the portfolio value associated with the contract and the value at this date of a referenced asset, index or mutual fund assumed to be less risky than the underlying of the policy-holder investment. In this article we consider that the term structure of interest rates is constant and we denote by r the interest rate.

4.1 Modelling contract and asset prices

We denote by S^2 the price process of the portfolio in which the initial amount, S_0^2 , of the policy-holder is invested. By definition S^1 is the risky asset price process which represents the guarantee. Generally, the initial value of this guarantee is different from the underlying portfolio of the policy-holder investment. To obtain a kind of homogeneity the initial guarantee is adjusted by a multiplicative factor such that we can consider $S_0^1 = S_0^2$.

In our model the market is not necessarily complete, if it is the case we work with a chosen risk-neutral measure. The usual way to do that is to use the measure implied by the market as explained by Björk (2004). Mathematically speaking the risk-neutral universe is given by the stochastic basis $(\Omega, \mathcal{A}, \mathcal{F}_{t,t \geq 0}, Q)$, where (Ω, \mathcal{A}, Q) is a probability space with the risk-neutral measure Q and $(\mathcal{F}_t, t \geq 0)$ is a filtration generated by the prices processes and represents the accumulated information with the passage of time.

Formally S^1 is assumed to obey the following stochastic differential equation:

$$\frac{dS_t^1}{S_t^1} = (r - \delta_1) dt + \sigma_1 dW_t$$

where W is a standard Q -Brownian motion \mathbf{Q} and δ_1 is a continuous dividend yield.

In the risk-neutral universe S^2 has the following dynamics

$$\frac{dS_t^2}{S_{t-}^2} = (r - \delta_2 - \lambda\kappa_2) dt + \rho\sigma_2 dW_t + \sigma_2\sqrt{1 - \rho^2} dZ_t + (Y_2 - 1) dN_t,$$

where Z is another standard Q-Brownian motion such that $d\langle W, Z \rangle_t = 0$ and N is a Poisson process with intensity: λ This jump process is independent from the two Brownian motions. Constant ρ is the instantaneous correlation between the two risky assets. The continuous dividend yield is denoted by δ_2 and jumps size, Y_2 , is an independent random variable with $\kappa_2 = E(Y_2 - 1)$.

We denote by Θ the time to maturity: remaining time up to expiry date. Thus an insured aged x when buying a contract will be $x + \Theta$ year old at expiry date. The payoff at this time writes,

$$\max\{S_{\Theta}^1, S_{\Theta}^2\} \mathbf{1}_{\{T_x > \Theta\}} = \left(S_{\Theta}^1 + [S_{\Theta}^2 - S_{\Theta}^1]^+ \right) \mathbf{1}_{\{T_x > \Theta\}}. \quad (19)$$

This payment can be seen as a long position in the guarantee portfolio and a long position in an exchange option written on the two involved assets, if the policy-holder is alive at the contract expiry date.

4.2 Valuation of the PEFG contract

If we assume independence between mortality risk and market risk, as it is usually done, the main problem to solve is to price an exchange option, if the underlying prices are geometric Brownian motions the solution is well known and is given by Margrabe's formula. However the problem is more difficult when jumps are introduced, and in this article we suggest a numerical solution based on Fourier analysis. At inception the contract value is obtained using

the arbitrage principle, which gives,

$$\begin{aligned} PEFG &= E_Q \left[e^{-r\Theta} \left(S_\Theta^1 + [S_\Theta^2 - S_\Theta^1]^+ \right) \mathbf{1}_{\{T > \Theta\}} \right] \\ PEFG &= \Theta p_x E_Q \left[e^{-r\Theta} \left(S_\Theta^1 + [S_\Theta^2 - S_\Theta^1]^+ \right) \right]. \end{aligned}$$

Because the discounted value of the gain process of S^1 is a Q martingale we can write,

$$E_Q \left[e^{-(r-\delta_1)\Theta} S_\Theta^1 \right] = S_0^1. \quad (20)$$

Finally the fair value of this contract at time zero is,

$$PEFG = \Theta p_x \left(S_0^1 e^{-\delta_1 \Theta} + C_0(S_0^1, S_0^2) \right), \quad (21)$$

where $C_0(S_0^1, S_0^2)$ is the price of an exchange option between S^1 and S^2 as defined by (29) and (30). Now let us explain how to efficiently price this option.

4.3 Pricing of the embedded exchange option

4.3.1 Dynamics

The first asset price is supposed to obey the following stochastic differential equation:

$$\frac{dS_t^1}{S_t^1} = (r - \delta_1) dt + \sigma_1 dW_t$$

where W is a standard Brownian motion in the risk-neutral universe \mathbf{Q} , σ_1 is the asset instantaneous volatility, and δ_1 is a constant continuous dividend rate. The quadratic variation per unit of time associated with this stochastic process is $V_1 = \sigma_1^2$.

The second asset S^2 pays out a dividend at a constant rate δ_2 . Its price is impacted by jumps. The dynamics of asset price S^2 are then represented by a jump diffusion:

$$\frac{dS_t^2}{S_t^2} = (r - \delta_2 - \lambda\kappa_2) dt + \rho\sigma_2 dW_t + \sigma_2\sqrt{1 - \rho^2} dZ_t + (Y_2 - 1) dN_t,$$

where Z is another standard Brownian motion such that $d\langle W, Z \rangle_t = 0$, and N is an independently distributed Poisson process with intensity λ . The constant ρ represents the instantaneous correlation between the diffusive components of both assets. The jump size Y_2 is a random variable independent from these processes, and $\kappa_2 = E(Y_2 - 1)$. Now, apart from the quadratic variation that results from the diffusive part, the jump component also contributes to V_2 , the overall quadratic variation per unit of time of the process.

Itô's lemma allows us to express the S^1 asset price's inverse as:

$$\begin{aligned} d\left(\frac{1}{S_t^1}\right) &= -\frac{1}{(S_t^1)^2} \left((r - \delta_1)S_t^1 dt + \sigma_1 S_t^1 dW_t \right) + \frac{\sigma_1^2 (S_t^1)^2}{(S_t^1)^3} dt \\ &= -\frac{1}{S_t^1} \left((r - \delta_1) dt + \sigma_1 dW_t \right) + \frac{\sigma_1^2}{S_t^1} dt \\ d\left(\frac{1}{S_t^1}\right) &= \frac{1}{S_t^1} (\sigma_1^2 - (r - \delta_1)) dt - \frac{\sigma_1}{S_t^1} dW_t. \end{aligned} \tag{22}$$

The covariation between $\frac{1}{S_t^1}$ and S_t^2 is:

$$d\left[\frac{1}{S_t^1}, S_t^2\right] = -\rho\sigma_1\sigma_2\frac{S_t^2}{S_t^1} dt.$$

We can now write the stochastic differential equation satisfied by S^2/S^1 :

$$\begin{aligned}
\frac{d(S_t^2/S_t^1)}{S_{t-}^2/S_t^1} &= \frac{S_t^1}{S_{t-}^2} \left(S_{t-}^2 d\left(\frac{1}{S_t^1}\right) + \frac{1}{S_t^1} dS_t^2 + d\left[\frac{1}{S_t^1}, S_t^2\right] \right) \\
&= S_t^1 d\left(\frac{1}{S_t^1}\right) + \frac{dS_t^2}{S_{t-}^2} + \frac{S_t^1}{S_{t-}^2} d\left[\frac{1}{S_t^1}, S_t^2\right] \\
&= (\sigma_1^2 - (r - \delta_1)) dt - \sigma_1 dW_t \\
&\quad + (r - \delta_2 - \lambda\kappa_2) dt + \rho\sigma_2 dW_t + \sigma_2\sqrt{1-\rho^2} dZ_t + \frac{\Delta S_t^2}{S_{t-}^2} dN_t \\
&\quad - \rho\sigma_1\sigma_2 dt \\
\frac{d(S_t^2/S_t^1)}{S_{t-}^2/S_t^1} &= \left((\delta_1 - \delta_2) + \sigma_1^2 - \lambda\kappa_2 - \rho\sigma_1\sigma_2 \right) dt \\
&\quad + (\rho\sigma_2 - \sigma_1) dW_t + \sigma_2\sqrt{1-\rho^2} dZ_t + (Y_2 - 1) dN_t. \tag{23}
\end{aligned}$$

Itô's lemma, again, allows us to express S^2/S^1 under \mathbf{Q} as:

$$\frac{S_t^2}{S_t^1} = \frac{S_0^2}{S_0^1} e^{\left((\delta_1 - \delta_2) + \frac{1}{2}(\sigma_1^2 - \sigma_2^2) - \lambda\kappa_2 \right) t + (\rho\sigma_2 - \sigma_1) W_t + \sigma_2\sqrt{1-\rho^2} Z_t + \sum_{i=1}^{N_t} \ln((Y_2)_i)}. \tag{24}$$

We can also recover Eq. (24) by computing the dynamics of both assets directly, as we have for asset S^1 :

$$S_t^1 = S_0^1 e^{(r - \delta_1 - \frac{1}{2}\sigma_1^2) t + \sigma_1 W_t}, \tag{25}$$

and, for asset S^2 :

$$S_t^2 = S_0^2 e^{\left(r - \delta_2 - \frac{1}{2}\sigma_2^2 - \lambda\kappa_2 \right) t + \rho\sigma_2 W_t + \sigma_2\sqrt{1-\rho^2} Z_t + \sum_{i=1}^{N_t} \ln((Y_2)_i)}. \tag{26}$$

4.3.2 Change of numéraire

To ease computation, we use the change of numéraire technique. Although it could be possible to use the general formula which involves the probability measures Q_S^1 and Q_S^2 , associated with the prices of the two risky assets:

We do not follow this way in our valuation. We will only use asset S^1 instead of the riskless asset B . After taking dividends into account, we can define the new probability measure Q_{S^1} with respect to the risk-neutral measure Q as follows:

$$\left. \frac{dQ_{S^1}}{dQ} \right|_{\mathcal{F}_T} = \frac{S_T^1}{S_0^1} e^{-(r-\delta_1)T} \quad (27)$$

From Eqs. (27) and (25), we get:

$$\left. \frac{dQ_{S^1}}{dQ} \right|_{\mathcal{F}_T} = e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1 W_T} = L_T^1,$$

where L^1 is a solution of the following stochastic differential equation:

$$\frac{dL_t^1}{L_t^1} = \sigma_1 dW_t.$$

Girsanov's theorem states that only the stochastic process W changes under the new measure Q_{S^1} : it is the process W' , such that $W'_t = W_t - \sigma_1 t$, which is now a standard Brownian motion under Q_{S^1} . In the $(\Omega, \mathcal{F}, \mathcal{F}_{t,t \geq 0}, Q_{S^1})$ universe, the dynamics of S^2/S^1 can be expressed as:

$$\frac{S_t^2}{S_t^1} = \frac{S_0^2}{S_0^1} e^{\left((\delta_1 - \delta_2) + \frac{1}{2}(\sigma_1^2 - \sigma_2^2) - \lambda \kappa_2 \right) t + (\rho \sigma_2 - \sigma_1) (W'_t + \sigma_1 t) + \sigma_2 \sqrt{1 - \rho^2} Z_t + \sum_{i=1}^{N_t} \ln \left((Y_2)_i \right)}$$

$$= \frac{S_0^2}{S_0^1} e^{\left((\delta_1 - \delta_2) + \rho\sigma_1\sigma_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) - \lambda\kappa_2\right)t + (\rho\sigma_2 - \sigma_1)W_t' + \sigma_2\sqrt{1-\rho^2}Z_t + \sum_{i=1}^{N_t} \ln\left((Y_2)_i\right)},$$

and, by taking $\tilde{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$:

$$\frac{S_t^2}{S_t^1} = \frac{S_0^2}{S_0^1} e^{\left((\delta_1 - \delta_2) - \lambda\kappa_2 - \frac{1}{2}\tilde{\sigma}^2\right)t + \tilde{\sigma}\tilde{W}_t + \sum_{i=1}^{N_t} \ln\left((Y_2)_i\right)} \quad (28)$$

where \tilde{W} is a standard Brownian motion under Q_{S^1} .

4.3.3 Generalized Fourier transform approach

Under the Q_{S^1} probability measure given by the change of numéraire (27), the exchange option price formula becomes:

$$\begin{aligned} C_0(S_0^1, S_0^2) &= E_Q[e^{-r\tau}(S_T^2 - S_T^1)^+] \\ &= E_{Q_{S^1}}\left[\frac{S_0^1}{S_T^1} e^{-\delta_1\tau}(S_T^2 - S_T^1)^+\right] \\ C_0(S_0^1, S_0^2) &= S_0^1 e^{-\delta_1\tau} E_{Q_{S^1}}\left[\left(\frac{S_T^2}{S_T^1} - 1\right)^+\right]. \end{aligned} \quad (29)$$

Now the payoff in the expectation under Q_{S^1} is similar to a European call option payoff written on an underlying S : $g(X_T) = (S_T - K)^+ = (S_0 e^{X_T} - K)^+$ with $S_0 = \frac{S_0^2}{S_0^1}$, $X_T = L_T$ where L is the exponential part of the dynamics of S^2/S^1 , and strike $K = 1$.

We can now implement the approach developed in Quittard-Pinon and Randrianarivony (2008a) by computing first the generalized Fourier trans-

form of the payoff function g . Straightforward computation gives:

$$\hat{g}(u) = \int_{\mathbb{R}} e^{-iux} g(x) dx = \frac{K e^{-iu \ln(K/S_0)}}{(-iu)(-iu+1)} = \frac{e^{-iu \ln(S_0^1/S_0^2)}}{(-iu)(-iu+1)},$$

where u is a complex number with the condition on the imaginary part $\Im u = \delta < -1$.

From Eq. (29), we get:

$$C_0(S_0^1, S_0^2) = S_0^1 e^{-\delta_1 \tau} \frac{1}{2\pi} \int_{\mathbb{R}+i\delta} E_{Q_{S^1}}[e^{iuX_T}] \hat{g}(u) du.$$

The expectation under the integral sign can be written as follows:

$$E_{Q_{S^1}}[e^{iuX_T}] = E_{Q_{S^1}}[e^{iuL_T}] = e^{-T\psi(u)}$$

where $\psi(u)$ is the characteristic exponent of the process L under Q_{S^1} .

The exchange option price can be expressed as:

$$\begin{aligned} C_0(S_0^1, S_0^2) &= S_0^1 e^{-\delta_1 \tau} \frac{1}{2\pi} \int_{\mathbb{R}+i\delta} e^{-T\psi(u)} \hat{g}(u) du \\ &= S_0^1 e^{-\delta_1 \tau} \frac{1}{2\pi} \int_{\mathbb{R}+i\delta} e^{-T\psi(u)} \frac{e^{-iu \ln(S_0^1/S_0^2)}}{(-iu)(-iu+1)} du \\ C_0(S_0^1, S_0^2) &= S_0^1 e^{-\delta_1 \tau} \frac{1}{2\pi} \int_{\mathbb{R}+i\delta} e^{iux} \frac{e^{-T\psi(u)}}{(-iu)(-iu+1)} du, \end{aligned} \quad (30)$$

where $x = \ln(S_0^2/S_0^1)$.

The $u \rightarrow u + i\delta$ change of variable yields:

$$C_0(S_0^1, S_0^2) = S_0^1 e^{-\delta_1 \tau} e^{-\delta x} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iux} \frac{e^{-T\psi(u+i\delta)}}{(-iu + \delta)(-iu + \delta + 1)} du.$$

Simplifying this expression gives:

$$C_0(S_0^1, S_0^2) = S_0^1 e^{-\delta_1 \tau} R(x, \delta) \times \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ixu} \Psi(u, \delta) du$$

where $R(x, \delta) = e^{-\delta x}$, and $\Psi(u, \delta) = \frac{e^{-T\psi(u+i\delta)}}{(-iu+\delta)(-iu+\delta+1)}$. This computation can be implemented directly through a Fast Fourier Transform (FFT) numerical scheme.

We only now need to specify the characteristic exponent ψ of L depending on its dynamics. Indeed, from Eq. (28), we have under measure Q_{S^1} :

$$L_t = \left((\delta_1 - \delta_2) - \lambda \kappa_2 - \frac{1}{2} \tilde{\sigma}^2 \right) t + \tilde{\sigma} \tilde{W}_t + \sum_{i=1}^{N_t} (J_2)_i$$

where $\tilde{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$, and \tilde{W} is a standard Brownian motion under Q_{S^1} .

More precisely, we have:

- if the jump size has a normal distribution, as in Merton (1976):

$$\psi(u) = -i \left((\delta_1 - \delta_2) - \lambda \kappa_2 - \frac{1}{2} \tilde{\sigma}^2 \right) u + \frac{1}{2} \tilde{\sigma}^2 u^2 - \lambda \left(e^{i\mu_J u - \frac{1}{2} \sigma_J^2 u^2} - 1 \right);$$

- if the jump size has a double exponential distribution, as in Kou (2002):

$$\psi(u) = -i\left((\delta_1 - \delta_2) - \lambda\kappa_2 - \frac{1}{2}\tilde{\sigma}^2\right)u + \frac{1}{2}\tilde{\sigma}^2u^2 - \lambda\left(\frac{p\eta_1}{\eta_1 - iu} + \frac{q\eta_2}{\eta_2 + iu} - 1\right).$$

4.4 Results

We have conducted a simulation with the following characteristics. The expiry date occurs when the insured is 75 year old. It can be seen that the effect of interest rate is weak. We, as announced in section 1, have used a Makeham law for the mortality and the estimates on American data obtained by Melnikov and Romaniuk (2006). We recall these estimates in Table 1.

Just give here the parameters related to the assets portfolio. We have standardized the initial values at contract inception to the values $S_0^1 = S_0^2 = 1$. The asset, which is considered as the guarantee, and whose price process is modeled by S^1 is assumed to deliver a continuous dividend with rate $\delta_1 = 0.01$. Its volatility is $\sigma_1 = 0.20$. The policy-holder invested portfolio price process is assumed to follow a Kou process with a Poisson arrival rate $\lambda = 0.5$. Jumps, $J = \ln(Y)$, modelled by independent identical random variables are supposed to follow a double exponential distribution whose density function reads:

$$f_J(y) = p\lambda_1 e^{-\lambda_1 y} 1_{y>0} + q\lambda_2 e^{\lambda_2 y} 1_{y\leq 0}$$

with $p \geq 0$, $q \geq 0$, $p + q = 1$, $\lambda_1 > 0$ et $\lambda_2 > 0$. Probability of an upwards jump is $p = 0.4$ and the two parameters associated with the double exponential law are fixed to $\lambda_1 = 10$ and $\lambda_2 = 5$.

We have rationally assumed that the guaranteed asset was less risky than

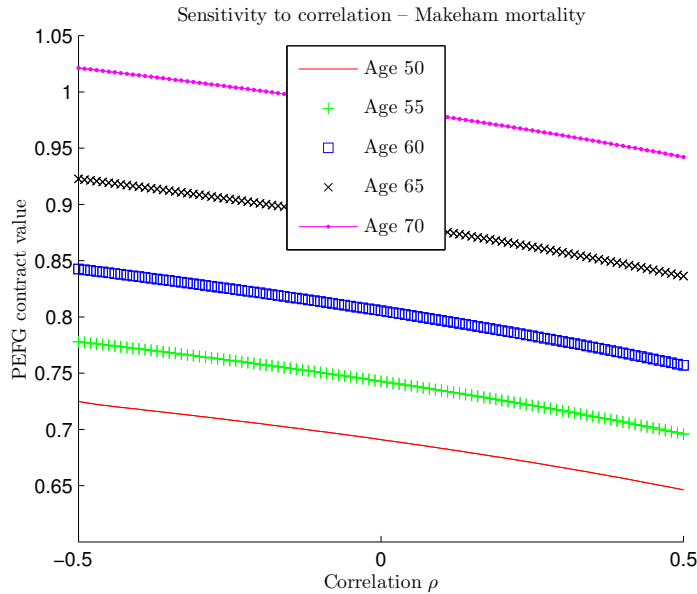


Figure 7 – Sensitivity of the PEFG contract value to the correlation ρ between the investment portfolio and the flexible guarantee.

the policy-holder invested portfolio. For a jump diffusion process, risk is not only those of the Brownian component, we have to take into account the risk emanating from the jumps. This risk is measured here by the jumps quadratic variation. We have chosen to fix σ_2 , the diffusive component of price process S^2 in such a way that the total quadratic variation is 1.5 times the quadratic variation of price process S^1 . This is a mean to keep the less risky demanded property for the flexible guarantee devoted to the first asset.

Figure 7 shows the sensitivity of the contract with respect to the correlation ρ between the two risky assets returns. We see the higher this correlation, the lower the contract value. Indeed if the two reference assets are correlated it is difficult to compensate a loss in the insured's portfolio by a gain in the flexible guarantee which probably will also suffer a loss. In these conditions the contract value will diminish.

Table 2 – Value of the PEFG contract with respect to insured age. Correlation level at $\rho = 0.25$.

Age (years)	Survival probability (at 75)	Exchange option	PEFG contract value
35	0.6058	0.9127	0.5529
40	0.6133	0.9592	0.5883
45	0.6235	1.0057	0.6270
50	0.6380	1.0510	0.6706
55	0.6597	1.0933	0.7213
60	0.6933	1.1295	0.7831
65	0.7473	1.1540	0.8623
70	0.8381	1.1524	0.9658

Furthermore, we also remark the later the subscription date the higher the contract value. For example for an insured aged between 65 and 70 year, the contract value is more expensive than this same policy for investors between 50 and 55. This phenomenon is confirmed by figure 9 where correlation is fixed to $\rho = 0.25$. Logically we find that the contract value rises with age. Table 2 gives contract values with respect to age at subscription, as well as the survival probability at 75 and the exchange option value embedded into the contract.

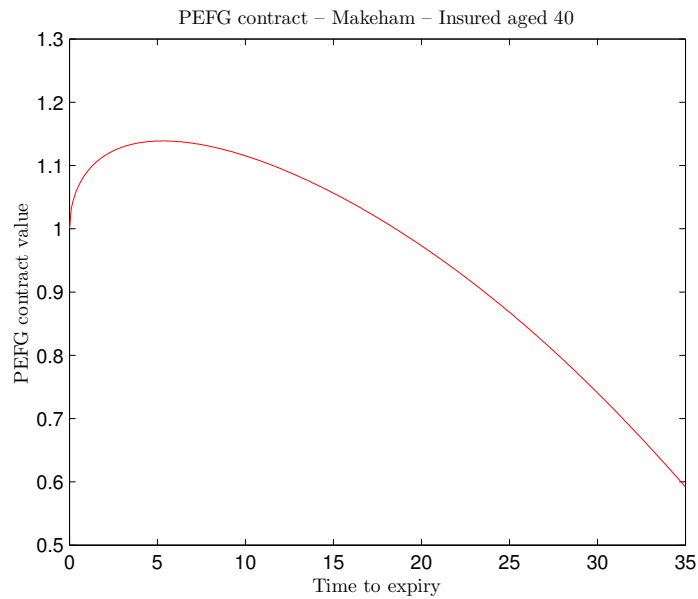


Figure 8 – Value of the PEFG contract with respect to time to expiry Θ . Insured aged 40. The correlation between the investment portfolio and the flexible guarantee is set at $\rho = 0.25$.

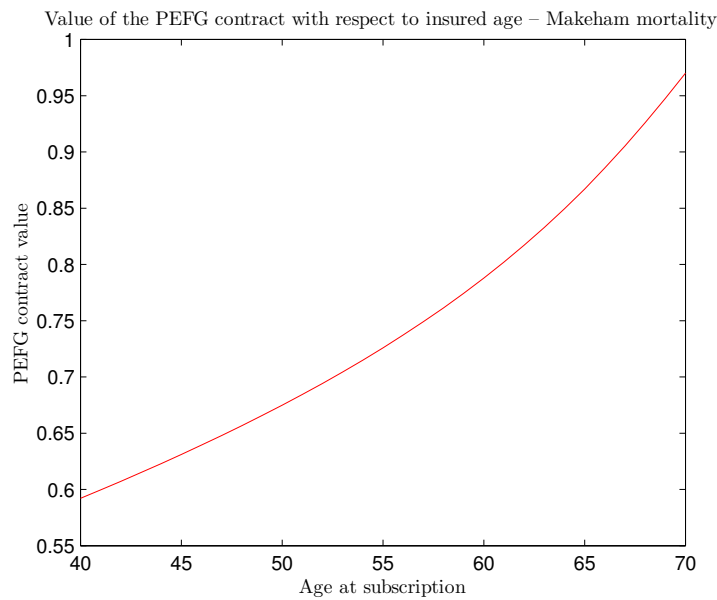


Figure 9 – Value of the PEFG contract with respect to age at subscription. The correlation between the investment portfolio and the flexible guarantee is set at $\rho = 0.25$.

5 Conclusion

In this paper we give a fast and efficient valuation for equity-linked pure endowment life insurance contracts with deterministic and stochastic guarantee in some non Gaussian financial markets whose price processes are described by stochastic volatility models and jumps. We emphasize the impact of this volatility for the GMMB contract while for pure endowment flexible guarantee we put forward the behaviour of its value with respect to correlation and show the higher the correlation the lower the contract value, whilst we analyse the impact of policy-holder age at subscription.

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