Understanding Stock Return Volatility

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October 13, 2008

Abstract

This article studies the effect of limited commitment on stock return volatility in a dynamic general-equilibrium economy populated by investors with heterogeneous beliefs. Due to heterogeneity of beliefs, investors disagree about the fundamentals, introducing an additional risk factor labeled sentiment risk. Limited commitment introduces an endogenous solvency constraint that scales up sentiment risk every time it binds. This is labeled solvency risk. I show that in equilibrium the market price of risk, which drives short-run stock return volatility, has three components: endowment risk, sentiment risk, and solvency risk. The solvency risk component in the market price of risk is novel and is the main contribution of the paper. I show that this model is better able to explain features of stock return volatility than conventional models, including volatility clustering, excess volatility, and asymmetry between stock market returns and volatility. Moreover, a multifactor consumption CAPM is obtained and the model predicts that the different components that drive volatility should be priced in the cross section of stock returns, in line with recent empirical evidence.

* I am specially grateful to Bernard Dumas for helpful discussions. I also thank Jerome Detemple, Darrell Duffie, Julien Hugonnier, Urban Jermann, Leonid Kogan and Karen Lewis for valuable suggestions. Versions of the paper where presented at University of Lausanne, Study Centre Gerzensee and University of Southern Denmark (Symposium on Stochastic Dynamic Models in Finance and Economics). I thank participants for their comments. Financial support from NCCR FinRisk is acknowledged with thanks.

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1 Introduction

Over the past few decades, research in financial economics has made a big effort to increase the understanding of the volatility patterns of stock-market returns. Indeed, good knowledge of return volatility is crucial for portfolio choice, risk management and derivatives asset pricing. Perhaps the most robust empirical regularity of stock return volatility is volatility clustering. As first noted by Mandelbrot (1963) referring to stock-market returns, "large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes". This persistence in volatility has lead to the development by Engle (1982) and Bollerslev (1986) of GARCH models, which have been very successful in modeling stock return volatility.

A second feature of stock-market return volatility that has received considerable attention is the apparent asymmetry in the relationship between stock-market return and volatility. Early influential studies by Black (1976) and Christie (1982) attribute the asymmetric return-volatility relationship to changes in financial leverage or debt-to-equity ratios. A drop in the value of the stock (negative return) increases financial leverage, which makes the stock riskier and increases its volatility. However, evidence reported by Schwert (1989) and Fligewski and Wang (2001) suggests that this cannot be the whole story. Another explanation for this asymmetry is the time-varying risk premium, or volatility-feedback effect, as discussed by French, Schwert and Stambaugh (1987) and Campbell and Hentschel (1992). If volatility is priced, an anticipated increase in volatility raises the required stock return, which generates an immediate stock price decline to allow for higher future returns. Bekar and Wu (2000) and Wu (2000) show empirically that the volatility-feedback effect dominates the leverage effect. But the empirical evidence is not conclusive. French, Schwert and Stambaugh (1987) and Campbell and Hentschel (1992) find the relation between volatility and expected return to be positive while Turner, Starz and Nelson (1989), Nelson (1991) and Glosten, Jangannathan and Runkle (1993) find the relation to be negative. Often the coefficient linking volatility to returns is statistically insignificant.

A third empirical finding is the fact that stock-return volatility contains different factors with different persistence. Engle and Lee (1999) argue that this may be interpreted as sensitivity of stock returns to shocks arriving at different frequencies. Several studies have found that multi-factor volatility models outperform single-factor specifications in explaining the behavior of stock-market return volatility. Importantly, multi-factor volatility models have also shown superior performance in the option pricing literature, as shown by Xu and Taylor (1994), Bates (2000) and Christophersen, Jacobs and Wang (2006). Moreover, in a recent contribution, Adrian and Rosenberg (2008) show that a CAPM, in which volatility factors are added to the market portfolio, compares favorably to some classic factor models in explaining the cross section of stock returns.

While the statistical knowledge of stock-return volatility is impressive, several questions remain regarding their economic explanation. For example, why does stock return volatility cluster? What are the reasons for the conflicting results regarding the relation between stock return volatility and stock expected returns? Why is stock return volatility asymmetric? Why is stock return volatility composed of several factors? What do these factors represent? Why do they matter in the cross section of stock returns? These questions have both a theoretical interest and far-reaching implications for portfolio choice, risk management and derivatives asset pricing.
article investigates these questions by introducing limited commitment in a general-equilibrium model in which investors have heterogeneous beliefs about the fundamentals. It is demonstrated that the interaction of heterogeneous beliefs with limited commitment introduces a new risk factor which is denoted solvency risk. This new risk factor is able to reproduce many of the stylized findings about stock return volatility and provides with potential answers to the questions raised above.

Two populations of investors receive specific endowments that are stochastic and commonly observable, but have incomplete (yet symmetric) information on their dynamics. In the continuous-time setting presented, both populations of investors deduce the volatility of the endowments, but must estimate their drifts. Due to heterogeneity of beliefs, each population of investors perform different inferences about the unobserved drifts of specific endowments. This leads to an additional risk factor denoted sentiment risk.

In every period, each population of investors receives its specific endowment and risk sharing is implemented through trade of contracts that specify future transfers of endowments between the two populations of investors. These contracts represent financial securities that pay the specific endowments as dividends. In the standard case where all investors can commit not to default on any prescribed endowment transfer, the optimal contracts achieve an efficient risk-sharing allocation. If, however, one population of investors faces limited commitment in the sense that it cannot commit not to default, then efficient risk sharing may be impeded by the possibility of ex-post default. In this context, contracts should be self-enforcing, such that for any period and state a solvency constraint that prevents default is imposed. In general, solvency constraints limit endowment transfers and, therefore, reduce the scope for risk sharing. The possibility of solvency constraints binding leads to an additional risk factor that scales up sentiment risk, and is labeled solvency risk.

The model yields the following results. First, the model produces stock return volatility clustering or GARCH effects. The market price of risk, which drives the short-run stock return volatility, has three components: endowment risk, sentiment risk and solvency risk. These three components are persistent, hence the model reproduces volatility clustering. Endowment risk is persistent because the "two-trees" feature of the model implies that endowment risk is proportional to the investors endowment shares of aggregate endowment, which fluctuate randomly between zero and one. Sentiment risk is persistent because it is closely related with the investors optimal consumption share of aggregate endowment, which fluctuates randomly between zero and one. Finally, solvency risk is persistent because the binding of solvency constraints is also persistent: periods of binding solvency constraints tend to be followed by periods of binding of solvency constraints, and periods of non-binding solvency constraints tend to be followed by periods of non-binding of solvency constraints.

Second, the model is consistent with multifactor volatility models. Both endowment risk and sentiment risk are associated with instantaneous shocks related with the idiosyncratic risk embedded in the Brownian motions present in the investors endowments. In contrast, solvency risk is associated with the binding of solvency constraints, and therefore it occurs at a lower frequency. Because the shadow price of solvency constraints alternates behavior between endogenous regimes of binding constraints and endogenous regimes of non-binding constraints, solvency risk exhibits

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2 In the current setup, risk sharing refers not only to specific-endowment risk sharing, but also to sentiment risk sharing.
different persistence. To my knowledge, this is the first paper which provides a theoretical foundation for the different components of stock return volatility and identifies potential sources for the different types of shocks occurring at different frequencies.

Third, the sign of the correlation between stock return volatility and stock expected return depends on the direction of disagreement of the population facing limited commitment. In periods when solvency constraints bind, the constrained investors experience a permanent increase in consumption, which is required in order to preclude default and switching to autarky. In order to restore the equilibrium in the financial markets, the interest rate must decrease. Due to no-arbitrage, the expected stock return must decrease by the same amount, implying that the binding of solvency constraints generates decreases in expected stock returns. In addition, solvency risk generates a positive premium in the market price of risk if the population of investors facing limited commitment is relatively optimistic, so that in this case the relationship between the instantaneous price of risk and stock return is negative. Conversely, solvency risk generates a negative premium in the market price of risk if the population of investors facing limited commitment is relatively pessimistic, so that in this case the relationship between the instantaneous price of risk and stock return is positive.

Fourth, the model provides a rationale to the empirical finding that volatility factors are priced in the cross section of stock returns. In the current model, stock return volatility is driven by three systematic risk factors: endowment, sentiment and solvency risks. These factors are shown to be priced across different stocks in an equilibrium consumption CAPM. Solvency risk increases the loading on the sentiment risk factor when solvency constraints bind, due to the scaling up of sentiment at times at which solvency constraints bind.

The main empirical prediction of the model is that stock return volatility should be predicted by disagreement and enforceability of contracts. On the first count, Athanassakos and Kalimipalli (2003) have shown empirically that there is a strong and positive relationship between analysts’ forecast dispersion and future return volatility. Concerning the second aspect, in a recent contribution Bae and Goyal (2008) demonstrate that variation in enforceability of contracts matters a great deal to how bank loans are structured and how they are priced. Although their analysis focuses on bank loans, their result that the pricing of financial contracts is greatly affected by the enforcement technology, is very much in line with the predictions of the current paper.

A second empirical prediction of the model is that the cross section of expected stock returns should be also predicted by disagreement and enforceability of contracts. Moreover, the cross section of expected stock returns should be predicted by the components of volatility associated with disagreement and enforceability of contracts, respectively. Adrian and Rosenberg (2008) show that the less persistent volatility component is significant in explaining the cross-section of stock returns because it relates to the pricing of skewness risk, which they interpret as a proxy for the tightness of financial constraints. In addition, they show that the more persistent volatility component is significant in explaining the cross-section of stock returns because it relates to the pricing of the business cycle. The tightness of financial constraints is related with the solvency constraints induced by limited commitment. In addition, since in the current model sentiment is correlated with the business cycle, their evidence is in line with the finding of this paper that sentiment drives stock return volatility.

A third implication of the model is that periods of asymmetric high volatility with low returns should be accompanied by decreases in liquidity. This occurs particularly in market crashes, as documented by Huang and Wang (2008).
On the technical side, I use both the martingale approach (Cox and Huang, (1989)) and the stopping time approach (El Karoui and Jeanblanc-Pique (1998) and Detemple and Serrat (2003)) in order to fully characterize the equilibrium state-price density, which allows me to examine short run stock return volatility in detail. It is demonstrated that the optimization problem of the population facing solvency constraints, taking prices as given, is equivalent to a stopping time problem, in which the surplus discounted expected utility of staying in the risk sharing agreement compared to choosing to default and switching to autarky for ever is optimally allocated over endogenous random time periods. These time periods correspond to epochs where solvency constraints are not binding. This is true for any candidate state price density, and in particular for the class of state price densities which are consistent with individual optimization and market clearing. These relationships allow me to find an equilibrium state price density which is consistent with both optimal enforcement of solvency constraints, as well as individual optimization and market clearing. To my knowledge, this is the first paper which fully characterizes equilibrium with heterogeneous beliefs and solvency constraints.3

The next section of the paper reviews additional literature that is related to this work. Section 3 presents the main setup of the model, including heterogeneity of beliefs, limited commitment and the investment opportunities. Section 4 determines the equilibrium in the economy. Section 5 describes the methodology used in order to solve for the equilibrium shadow price of solvency constraints. Section 6 implements the complete financial market equilibrium, characterizing stock return volatility and stock expected returns. Section 7 contains the conclusions. All the mathematical derivations are contained in the appendix.

2 Review of the literature

There have been some attempts to explain theoretically the behavior of the stock return volatility. Veronesi (1999) constructs a model with regime shifts in the endowments in which investors willingness to hedge against their own uncertainty on the true regime generates overreaction to good news in bad times and volatility clustering. In contrast to that paper, I assume that the exogenous state variables are not subject to regimes, neither do they exhibit mean-reversion. For instance, the current paper obtains volatility clustering in equilibrium even when all the exogenous state variables are geometric Brownian motions.

Similarly, Li and Muzere (2007) build an equilibrium two-country, two-good model with log-arithmetic utility (myopic) investors with exogenous disagreement and heterogeneous subjective discount rates. They claim that "the model can also generate the clustering of the volatility of foreign exchange and stocks if the differences of beliefs are clustering". In contrast, in this paper the clustering of volatility is not conditional on the clustering of disagreement. In particular, the most general (constant) disagreement process is chosen, and equilibrium volatility clustering as well as other volatility empirical regularities are shown to hold.

In addition McQueen and Vorkink (2004) are able to reproduce GARCH volatility using a preference-based model with time-varying sensitivity to news. In a related paper, Vanden (2005) develops a model where the representative agent exhibits a utility function with several risk aversion regimes, which in equilibrium leads to volatility regimes and volatility clustering. In contrast to

3Moreover, to my knowledge this is the first paper which fully characterizes equilibrium with solvency constraints in a continuous time setting, which is equivalent to a generalization of current models (which assume a finite number of states in a markov chain) to an infinite state-space.
these models, I assume standard constant relative risk aversion preferences and volatility clustering occurs for different possible levels of risk aversion.

In a recent contribution, Gallmeyer and Hollifield (2008) propose a model with heterogeneous beliefs and short sale constraints. They are able to show some effects of heterogeneous beliefs and of short sales constraints on equilibrium volatility at a fixed point in time through extensive simulations. However, their constraints are imposed exogenously, while the rich dynamics obtained in this paper come from the endogenous solvency constraints and binding regimes which arise naturally from limited commitment.

The model presented in this paper is close to a strand in the literature in which investors have incomplete (but symmetric) information about a relevant fundamental, and due to heterogeneous beliefs they disagree about their expected value of that fundamental. Important contributions related to this literature are Gallmeyer (2002) Scheinkman and Xiong (2003), Basak (2005), and Dumas, Kurshev and Uppal (2008). This paper can be seen as an extension of these models, by introducing in addition limited commitment and endogenous solvency constraints.

The model presented is also related to the literature in limited risk sharing and participation constraints (Kehoe and Levine (1993), Kocherlakota (1996) and Zhang (1997)). More recently, Alvarez and Jermann (2000, 2001) showed that these participation constraints can be modelled as endogenous portfolio constraints, which they denote solvency constraints. Besides extending this strand of the literature for the case of heterogeneous beliefs, I solve this problem in continuous time for the first time, thus generalizing it to the case of infinite possible states. Moreover, I show that the problem can be solved using the stopping time approach, which is insightful since it shows that the allocation of investors facing solvency constraints involves the replication of a sequence of American put options on the surplus utility of staying in the risk sharing agreement compared to choosing to default and switching to autarky. In this aspect, the methodology employed is close to that used in the literature on asset pricing with liquidity constraints (El Karoui and Jeanblanc-Pique (1998) and Detemple and Serrat (2003)).

3 The Setup

I consider a continuous-time "two-trees" pure-exchange economy with infinite horizon. There is a single consumption good which serves as the numeraire. There are two types of investors who are heterogeneous in endowments, beliefs, and enforcement technology. Heterogeneity in endowments and beliefs provide incentives for trade in the financial markets, which is driven by risk sharing motives and sentiment, respectively. In contrast, heterogeneity in the enforcement technology provides incentives to stop trading in the financial markets in particular situations when it is anticipated that a counterparty with poor enforcement technology would be tempted to default on current trades.

This section describes the key features of the model. I adopt a notation close to the one used in the paper by Dumas, Kurshev and Uppal (2008)

3.1 Information structure and investors perceptions

There are two groups of investors, and each group may be characterized by a representative investor $i = \{A, B\}$. Both investors commonly observe the investor-specific endowments, but have
incomplete (yet symmetric) information on its dynamics. The investor-specific endowments $\delta_{i,t}$ follow

$$d\delta_{A,t} = \delta_{A,t} \left[ \mu_{\delta_{A}} dt + \sigma_{\delta_{A}} dZ_{\delta_{A},t} \right] \quad \text{(1)}$$

$$d\delta_{B,t} = \delta_{B,t} \left[ \mu_{\delta_{B}} dt + \sigma_{\delta_{B}} dZ_{\delta_{B},t} \right] \quad \text{(2)}$$

where $Z_{\delta_{A},t}$ and $Z_{\delta_{B},t}$ are independent standard Brownian motions under the objective probability measure, which governs empirical realizations of the process.

Upon observations of $(\delta_{A,t}, \delta_{B,t})$ both investors can deduce $(\delta_{A}, \delta_{B})$ from the quadratic variation of the endowments, but because they don’t observe the Brownian motions, they can only draw inferences about $(\delta_{A}, \delta_{B})$. It is assumed that these inferences are different for each investor. An explicit analysis of the inference problems is not provided, since this is not necessary for the determination and characterization of equilibrium under heterogeneity of beliefs.

These stochastic differential equations can be written in terms of processes that are Brownian under subjective probability measures. Consider a two-dimensional process $W^{B} = \left( W^{B}_{\delta_{A}}, W^{B}_{\delta_{B}} \right)$ that is Brownian under the probability measure that reflects the expectations of investor $B$. Denote by $(\hat{\mu}^{B}_{\delta_{A}}, \hat{\mu}^{B}_{\delta_{B}})$ the inference of $(\mu_{\delta_{A}}, \mu_{\delta_{B}})$ by investor $B$. Then we can write

$$d\hat{\delta}_{A,t} = \delta_{A,t} \left[ \hat{\mu}^{B}_{\delta_{A}} dt + \sigma_{\delta_{A}} dW_{\delta_{A},t}^{B} \right] \quad \text{(3)}$$

$$d\hat{\delta}_{B,t} = \delta_{B,t} \left[ \hat{\mu}^{B}_{\delta_{B}} dt + \sigma_{\delta_{B}} dW_{\delta_{B},t}^{B} \right] \quad \text{(4)}$$

Due to observational equivalence with the processes under the objective probability measure, it follows that

$$dW_{\delta_{A},t}^{B} = dZ_{\delta_{A},t} - \frac{\hat{\mu}^{B}_{\delta_{A}} - \mu_{\delta_{A}}}{\sigma_{\delta_{A}}} dt \quad \text{(5)}$$

$$dW_{\delta_{B},t}^{B} = dZ_{\delta_{B},t} - \frac{\hat{\mu}^{B}_{\delta_{B}} - \mu_{\delta_{B}}}{\sigma_{\delta_{B}}} dt \quad \text{(6)}$$

A similar two-dimensional process $W^{A} = \left( W^{A}_{\delta_{A}}, W^{A}_{\delta_{B}} \right)$ that is Brownian under the subjective probability measure of investor $A$ could be defined. Denote by $(\hat{\mu}^{A}_{\delta_{A}}, \hat{\mu}^{A}_{\delta_{B}})$ the inference of $(\mu_{\delta_{A}}, \mu_{\delta_{B}})$ by investor $A$. Then, the relation between the Brownians under different subjective measures is:

$$dW_{\delta_{A},t}^{B} = dW_{\delta_{A},t}^{A} - \overline{\mu}_{\delta_{A}} dt \quad \text{(7)}$$

$$dW_{\delta_{B},t}^{B} = dW_{\delta_{B},t}^{A} - \overline{\mu}_{\delta_{B}} dt \quad \text{(8)}$$

where

$$\overline{\mu}_{\delta_{A}} = \frac{\hat{\mu}^{B}_{\delta_{A}} - \hat{\mu}^{A}_{\delta_{A}}}{\sigma_{\delta_{A}}} \quad \text{(9)}$$
The processes \((\bar{\mu}_{\Delta A}, \bar{\mu}_{\Delta B})\) parametrize investor’s disagreement on the expected growth rate of the endowments, normalized by their risk. When \(\bar{\mu}_{\Delta} > 0\) investor \(A\) is pessimistic with respect to investor \(B\) about the endowment corresponding to investor \(i\), and conversely. While disagreement \((\bar{\mu}_{\Delta A}, \bar{\mu}_{\Delta B})\) could be described by a wide class of processes, I follow Kogan, Ross, Wang and Westerfield (2006) in assuming the most general case where they are considered as constants.\(^4\)

In the current model, no agent knows the true state of the economy. Hence, the objective measure is not defined on either agent’s \(\sigma\)-algebra and it can be ignored for the purpose of calculating the equilibrium. Investor \(B\) probability measure is used as the reference measure. From Girsanov theorem and equations (7) and (8), we can determine that the change from the subjective probability measure of investor \(B\) to the subjective probability measure of investor \(A\) is given by the exponential martingale \(t\), which follows:\(^5\)

\[
d\eta_t = -\eta_t (\bar{\mu}_{\Delta A} dW_{\Delta A,t}^B + \bar{\mu}_{\Delta B} dW_{\Delta B,t}^B)
\]  

The role of \(\eta_t\), which following Dumas, Kurshev and Uppal (2008) I denote sentiment risk, is key in the properties of the equilibrium. This state variable determines how investor \(A\) over or underestimates the probability of a state relative to investor \(B\). Importantly, its volatility is driven by disagreement \((\bar{\mu}_{\Delta A}, \bar{\mu}_{\Delta B})\).

For tractability I make the following change of variables. Instead of \(\delta_{B,t}\) I consider the ratio of endowments \(\frac{\delta_{B,t}}{\delta_{A,t}}\). A straightforward application of Ito Lemma shows that the dynamics of \(\eta_t\) are given by

\[
d\eta_t = \eta_t \left[ (\bar{\mu}^B_{\Delta} - \bar{\mu}^B_{\Delta A}) dt - \sigma_{\Delta A} dW_{\Delta A,t}^B + \sigma_{\Delta B} dW_{\Delta B,t}^B \right]
\]  

The markovian system made of equations (3), (11) and (12) completely characterizes the evolution of the exogenous state variables \(\{\delta_A, r, \eta\}\) under the subjective measure of investor \(B\), with diffusion matrix:

\[
\begin{bmatrix}
\frac{dW_{\Delta A,t}^B}{\delta_{A,t}} & \frac{dW_{\Delta B,t}^B}{\delta_{B,t}} \\
-\sigma_{\Delta A} & 0 \\
-\eta_t \bar{\mu}_{\Delta A} & -\eta_t \bar{\mu}_{\Delta B}
\end{bmatrix}
\]  

3.2 Limited commitment

In the setup described, the investors trade in the financial market in order to diversify specific-endowment risk and due to disagreement in the fundamentals. The existing literature on pure exchange economies with heterogeneous beliefs assumes that the enforcement technology prevents any investor from defaulting in their promised endowment claims, implying that they can fully

\(^{4}\)This paper obtains volatility clustering in general equilibrium without exogenous mean reversion or exogenous regimes in any of the state variables. With this aim, the most general cases where assumed, but more involved disagreement or endowment processes would not affect the main results of this paper.

\(^{5}\)For any uncertain event \(e\) at future time \(u\) we have that \(\mathbb{E}_t^B (1_{e_u}) = \mathbb{E}_t^B \left( \frac{\mathbb{E}_u^B (1_{e_u})}{\mathbb{E}_u^B (1_{e_u})} \right)\)
commit to honour their obligations. However, if contracts written by one sector of the economy were not fully enforceable, these investors could default provided this makes them better off, according to the value of the outside option.

I consider an economy where, if agents facing limited commitment default on their promised endowment claims, they can be punished by seizing all the assets that they may own, but they cannot be punished by garnishing their specific-endowment. This leads to a competitive equilibrium with solvency constraints in the spirit of Alvarez and Jermann (2000, 2001). Although not modeled explicitly, the nature of this punishment implicitly assumes that, from the point of view of the lender, the cost of default is always higher than the expected benefit in terms of future risk sharing opportunities from letting the investor which chooses to default stay in the risk sharing agreement. It is worth noting that because exclusion of the financial markets is the worst possible punishment, it is the one which maximizes welfare, because it minimizes the binding of solvency constraints, leading to the maximum possible risk sharing. In other words, I focus in solvency constraints which are tight enough to prevent default but allow as much risk sharing as possible. Alvarez and Jermann (2000, 2001) denote this property as not too tight solvency constraints. The main difference with respect to their paper is that I introduce heterogeneous beliefs, and the results in this paper depend crucially in the interaction of limited commitment with heterogeneous beliefs. Another difference in the limited commitment formulation used in this paper is that I assume an heterogeneous enforcement technology, such that only one group of investors is unable to fully commit.\footnote{This can be seen as an intermediate case between the benchmark case of full commitment, and the extreme case where every investor faces limited commitment. If we let investors from both groups face limited commitment, then solvency risk would be enhanced and its effects in volatility would be increased.}

It is assumed that investors from Group $B$ face an enforcement technology which prevents them to default in every possible state, such that they are able to fully commit. In contrast, investors from Group $A$ face an enforcement technology which implies they may default in some states, such that their degree of commitment is limited. In particular, \emph{ex-post} an investor from Group $A$ would decide to default on his promised endowment claim and switch to autarky forever when his discounted expected utility of doing so is higher to that of staying in the risk sharing agreement. Hence, because solvency constraints are not too tight, \emph{ex-ante} no investor will buy the claims (lend) from an investor from Group $A$ above a certain amount, above which it is known that he would default and stop participating in the risk sharing agreement: limited commitment generates endogenous solvency constraints. In this context, solvency constraints which ensure that the discounted expected utility of staying in the risk sharing agreement is at no time or state lower than the discounted expected utility of choosing to default and switching to autarky for ever are imposed. In particular, assuming the instantaneous utility $U$ of consumption $c_{A,t}$ is given by $U(c_{A,t})$, with discount factor $\rho$ and infinite horizon, the solvency constraint faced by the representative investor of population $A$ (here after Investor $A$) is given by:

\begin{equation}
\mathbb{E}^A_t \int_t^\infty e^{-\rho(u-t)} U(c_{A,u}) \, du \geq \mathbb{E}^A_t \int_t^\infty e^{-\rho(u-t)} U(\delta_{A,u}) \, du
\end{equation}

This equation states that investor $A$ is limited to consumption policies that ensure that he remains solvent, in the sense that his incentives are aligned with staying in the risk sharing agreement.
at all times and states. The analysis presented shows that in order to fulfill this solvency constraint, Investor A needs to replicate a sequence of American-style contingent claims written on the utility surplus of staying in the risk sharing agreement compared to switching to autarky for ever, which are exercised every time the solvency constraint binds.

As noted, the solvency constraint is in fact a direct constraint on the consumption profile of Investor A, as opposed to an exogenous portfolio constraint. The implications for equilibrium are therefore very different from those of models with exogenous short sales constraints. Solvency constraints induce Investor A to partially substitute risk sharing from trading in financial securities at each time period, by intertemporal risk sharing through shifting consumption from the present to the future (saving) uniformly across states of nature. In the early stages Investor A defers consumption in order to build protection for future situations where he would not be able to sell financial claims to his future endowment (borrowing). At later stages of the life cycle he compensates by consuming more than in the standard case with full commitment, when solvency constraints bind. Deferral of consumption is set such that solvency constraints are satisfied at every future period.

The probability of solvency constraints binding is decreasing both in the ratio of endowments \( r \) and in the ratio of probability measures \( \eta \). In states where \( r \) is high, Investor A is relatively poor compared to Investor B. Therefore in these states the value of the option to switch to autarky is low, and the probability of solvency constraints binding is low. It will soon become apparent that in states where \( \eta \) is high, the optimal consumption share of aggregate endowment of Investor A is relatively high compared to that of Investor B. Therefore in these states the value of staying in the risk sharing agreement is high, and the probability of solvency constraints binding is low.

The solvency constraint precludes default by inducing a permanent increase in investor A optimal consumption share of aggregate endowment every time it binds. Successive bindings of the solvency constraint will be accompanied by successive permanent increases in consumption. As such, solvency constraints induce a segmentation of the state space which prevents investors from equalizing marginal utilities to state price ratios uniformly across states of nature.

### 3.3 Investment Opportunities

Since there are two Brownian motions in the economy \((W^B_{\delta A,t}; W^B_{\delta B,t})\), three securities which are not linearly dependent are required to implement the equilibrium. I assume trading may take place continuously in a riskless instantaneous bank deposit and two risky securities. The bank deposit has value \( B_t \), with dynamics

\[
\frac{dB_t}{B_t} = i_t dt - dQ_t
\]

where \( i_t \) is the instantaneous interest rate and \( Q_t \) is a strictly positive singular process which is associated with limited commitment. The risky securities with prices \( S^A_{A,t} \) and \( S^B_{B,t} \) are contingent claims on each investor’s endowments \((\delta_{A,t}, \delta_{B,t})\), respectively. Therefore, their gain processes under the reference measure follow

\[
dS^A_{A,t} + \delta_{A,t} dt = S^A_{A,t} \left[ \tilde{\mu}^B_{S^A,t} dt - dQ_t + \sigma^B_{S^A,t} dW^B_{\delta^A,t} + \sigma^B_{S^A,t} dW^B_{\delta^B,t} \right]
\]

\[
dS^B_{B,t} + \delta_{B,t} dt = S^B_{B,t} \left[ \tilde{\mu}^B_{S^B,t} dt - dQ_t + \sigma^B_{S^B,t} dW^B_{\delta^A,t} + \sigma^B_{S^B,t} dW^B_{\delta^B,t} \right]
\]
where $\hat{\mu}_{S,t}^i$ is Investor $i$ belief about the instantaneous expected stock return and $Q_t$ is the singular process introduced before. This formulation for the dynamics of financial securities is similar to that in Detemple and Serrat (2003), except that this model has two risky securities associated with the "two-trees" feature of the endowments. Moreover, in the current model the appearance of singular components obeys to limited commitment as opposed to wealth constraints.

All investors agree about the diffusion coefficients on the stock gain processes. By agreeing on the traded stock prices, the investor-specific expected stock returns are linked via

$$
\begin{align*}
\hat{\mu}_{S,t}^A - \hat{\mu}_{S,t}^B &= \sigma_{S,t}^A \bar{\pi}_{S,t} + \sigma_{S,t}^B \bar{\pi}_B \\
\hat{\mu}_{S,t}^B - \hat{\mu}_{S,t}^A &= \sigma_{S,t}^A \bar{\pi}_A + \sigma_{S,t}^B \bar{\pi}_B
\end{align*}
$$

(18)

The perceived state price density under the reference measure $\xi_t^B$ is given by

$$
d\xi_t^B = -\xi_t^B \left[ dB_t + \kappa_{A,t}^B dW_{A,t}^B + \kappa_{B,t}^B dW_{B,t}^B \right]
$$

(20)

where $\left(\kappa_{A,t}^B, \kappa_{B,t}^B\right)^T$ is the vector of market prices of risk under the measure of investor $B$, satisfying

$$
\begin{bmatrix}
\hat{\mu}_{A,t}^B - i_t \\
\hat{\mu}_{B,t}^B - i_t
\end{bmatrix}
= \begin{bmatrix}
\sigma_{A,t}^B & \sigma_{A,t}^B \\
\sigma_{B,t}^B & \sigma_{B,t}^B
\end{bmatrix}
\begin{bmatrix}
\kappa_{A,t}^B \\
\kappa_{B,t}^B
\end{bmatrix}
$$

(21)

It follows that the investor-specific market prices of risk are linked via

$$
\begin{bmatrix}
\kappa_{A,t}^B \\
\kappa_{B,t}^B
\end{bmatrix} - \begin{bmatrix}
\kappa_{A,t}^B \\
\kappa_{B,t}^B
\end{bmatrix} = \begin{bmatrix}
\bar{\pi}_A \\
\bar{\pi}_B
\end{bmatrix}
$$

(22)

The endogenous price system $\left(i_t, Q_t, \hat{\mu}_{S,t}^A, \hat{\mu}_{S,t}^B, \sigma_{S,t}^A, \sigma_{S,t}^B, \sigma_{A,t}^A, \sigma_{A,t}^B, \sigma_{B,t}^A, \sigma_{B,t}^B\right)$ is determined in equilibrium.$^7$

### 4 Individual Optimization Problem and Equilibrium

In this section I first use the martingale approach in order to solve for the optimal consumption problem of each investor. Then, under the complete markets setting I compute the unique state price density consistent with such optimization and market clearing, but assuming a general shadow price of solvency constraints. I conclude this section by computing the equilibrium shadow price of solvency constraints using the stopping time approach, which provides a full characterization of the equilibrium.

#### 4.1 Preference of agents and optimization problems

The main goal of this paper is to analyze the equilibrium implications of the interaction of heterogeneous beliefs and limited commitment in stock return volatility. Differences in risk aversion and in the rate of impatience are therefore not the main focus. Thus, the model is restricted to the case of constant relative risk aversion utility function with the same risk aversion $1 - \alpha$ and with the same rate of impatience $\rho$ for both investors.

$^7$Determining the equilibrium under the measure of Investor $A$ would be then straightforward by using the relations in (18), (19) and (22).
Under the reference measure of investor $B$, investor $A$ faces the optimization problem:\(^8\)

$$\sup_{c_{A,t}} \mathbb{E}_0^B \int_0^\infty \eta_u e^{-\rho u} \frac{1}{\alpha} c_{A,u}^\alpha du$$  \hspace{1cm} (23)$$

subject to:

$$\mathbb{E}_0^B \int_0^\infty \xi_u^B (c_{A,u} - \delta_{A,u}) \, du \leq 0$$  \hspace{1cm} (24)$$

$$V_t = \mathbb{E}_t^B \int_t^\infty \frac{\eta_u}{\eta_t} e^{-\rho (u-t)} \frac{1}{\alpha} (c_{A,u}^\alpha - \delta_{A,u}^\alpha) \, du \geq 0, \text{ for all } t$$  \hspace{1cm} (25)$$

where we assume $\eta_0 = 1$. The appendix shows that optimal consumption of investor $A$ satisfies:

$$c_{A,t} = \left[ \left( \frac{1}{1 + \beta_t} \right) \frac{\lambda_A}{\eta_t} e^{\rho t} \xi_t^B \right]^{-\frac{1}{\alpha-1}}$$  \hspace{1cm} (26)$$

where $\lambda_A$ is the constant shadow price of the static budget constraint (24) and $\beta_t$ is the stochastic shadow price of the dynamic solvency constraint (25). Since the solvency constraint must be satisfied at any time and in any state, the multiplier $\beta_t$ is a stochastic process. Furthermore, $\beta_t$ must be predictable, non-decreasing, and satisfy the complementary slackness condition:

$$\mathbb{E}_0^B \int_0^\infty V_t d\beta_t = 0$$  \hspace{1cm} (27)$$

The latter implies that when the solvency constraint is slack ($V_t > 0$) the multiplier remains constant ($d\beta_t = 0$), and when the multiplier increases ($d\beta_t > 0$) the solvency constraint is binding ($V_t = 0$).

Investor $B$ faces a similar problem but without solvency constraints:

$$\sup_{c_{B,t}} \mathbb{E}_0^B \int_0^\infty e^{-\rho u} \frac{1}{\alpha} c_{B,u}^\alpha du$$  \hspace{1cm} (28)$$

subject to:

$$\mathbb{E}_0^B \int_0^\infty \xi_u^B (c_{B,u} - r_u \delta_{A,u}) \, du \leq 0$$  \hspace{1cm} (29)$$

The optimal consumption of investor $B$ satisfies:

$$c_{B,t} = \left( \lambda_B e^{\rho t} \xi_t^B \right)^{-\frac{1}{\alpha-1}}$$  \hspace{1cm} (30)$$

where $\lambda_B$ is the constant shadow price of the static budget constraint (29).

\(^8\)For ease of comparison with the existing literature on limited commitment, I assume the solvency constraint is imposed in the optimization problem of the constrained investor. If the solvency constraint was imposed in the optimization problem of the unconstrained investor the results would be identical, since both cases are equivalent to solving the planner problem, as shown by Alvarez and Jermann (2000).
4.2 Equilibrium for a general shadow price of solvency constraints

The aggregate resource constraint implies that in equilibrium:

\[ c_A; t + c_B; t = \delta_A; t (1 + r_t) \]  

(31)

Plugging optimal consumption (26) and (30) we get:

\[ \left( \frac{1}{1 + \beta_t} \right) \frac{\lambda_A}{\eta_t} e^{\rho t} \xi^B_t \left( \frac{1}{1 - \alpha} \right) + \left( \frac{\lambda_B}{\lambda_A} \xi^B_t \right) \left( \frac{1}{1 - \alpha} \right) = \delta_A; t (1 + r_t) \]  

(32)

Solving this equation we obtain the equilibrium state price density

\[ \xi^B_t = e^{-\rho t} \frac{1}{\lambda_B} \left( 1 + \left( \frac{\lambda_B}{\lambda_A} (1 + \beta_t) \eta_t \right) \right)^{1 - \alpha} \left[ \delta_A; t (1 + r_t) \right]^{-(1 - \alpha)} \]  

(33)

where \( \beta_t \) is a general shadow price of solvency constraints (its equilibrium value is determined in the next subsection).

Optimal consumption is then given by

\[ c_A; t = \left[ \omega (\eta_t, 1 + \beta_t) \right] \delta_A; t (1 + r_t) \]  

(34)

\[ c_B; t = [1 - \omega (\eta_t, 1 + \beta_t)] \delta_A; t (1 + r_t) \]  

(35)

where:

\[ \omega (\eta_t, 1 + \beta_t) = \frac{\left[ \frac{\lambda_B}{\lambda_A} (1 + \beta_t) \eta_t \right]^{1 - \alpha}}{1 + \left[ \frac{\lambda_B}{\lambda_A} (1 + \beta_t) \eta_t \right]^{1 - \alpha}} \]  

(36)

is the optimal consumption share of aggregate endowment of Investor A. Optimal consumption is linear in the aggregate endowment \( \delta_A; t (1 + r_t) \) because both groups have the same risk aversion. However the share of consumption allocated to Investor A, \( \omega (\eta_t, 1 + \beta_t) \), is stochastic and driven both by sentiment risk \( \eta_t \) and solvency risk, represented by the shadow price of solvency constraints \( \beta_t \). Furthermore, notice that the share of aggregate consumption allocated to the constrained investor \( \omega (\eta_t, 1 + \beta_t) \) is increasing in both \( \eta_t \) and \( \beta_t \). Moreover, since \( \beta_t \) is constant when solvency constraints are not binding and it increases when solvency constraints bind, it is already clear from the term \( \eta_t (1 + \beta_t) \) present in \( \omega \) and \( \xi^B_t \) that when solvency constraints bind sentiment risk is scaled up.

It will be useful for the next sections to disentangle the optimal share of aggregate consumption \( \omega (\eta_t, 1 + \beta_t) \) into a sentiment component involving \( \eta_t \) (exogenous state variable) and a solvency constraint component associated with \( \beta_t \) (endogenous state variable). This is done in the next proposition.\(^9\)

---

\(^9\)Proposition 1 assumes \(-\alpha\) is a positive integer, and by disentangling sentiment and solvency risks provides with simplification in the solution methodology described in section 5. However, it is worth mentioning that for the case where \(-\alpha\) is either negative (risk loving) or not an integer, a solution can still be obtained, although the method gets slower.
Proposition 1. Denote by $\theta$ the share of aggregate consumption of investor A that would obtain in the absence of sentiment risk ($\eta_t = 1$ for all $t$). That is, let

$$\theta (1 + \beta_t) = \omega (1, 1 + \beta_t) = \frac{\lambda_A \eta_t (1 + \beta_t)^{1-\alpha}}{1 + \left[ \frac{\lambda_B}{\lambda_A} (1 + \beta_t) \right]^{1-\alpha}}$$

Then, if we assume that $-\alpha$ is a positive integer, the effects of sentiment risk $\eta_t$ and the effect of solvency risk $\theta (1 + \beta_t)$ can be disentangled from $\omega (\eta_t, 1 + \beta_t)$ in the following manner:

$$\omega (\eta_t, 1 + \beta_t) = \theta (1 + \beta_t) \left( \sum_{j=0}^{-\alpha} \left( \begin{array}{c} -\alpha \\ j \end{array} \right) \left[ 1 - \theta (1 + \beta_t) \right]^j \sum_{k=0}^{j} \left( \begin{array}{c} j \\ k \end{array} \right) (-1)^k \eta_t^{\frac{k}{\alpha}} \right)^\frac{1}{\alpha}$$

Proof: See the appendix.

Given the constant shadow prices $\lambda_A$ and $\lambda_B$, and given the exogenous processes $\{\delta_{A,t}, r_t, \eta_t\}$ I have now characterized the complete-market equilibrium, except for the shadow price of solvency constraints $\beta_t$. This is done in the next subsection.

4.3 Equilibrium shadow price of solvency constraints

The previous subsection has shown that the equilibrium state price density and optimal consumption depend not only on the exogenous state variables, but also on the endogenous shadow price of solvency constraints $\beta_t$. In equilibrium, the solvency constraints must hold in such a way that consumption is optimal taking into account the effect of solvency constraints in the state price density. This yields to the following stopping time characterization.

Proposition 2. Denote by $\Delta_t (q)$ the instantaneous surplus utility of investor A from staying in the risk sharing agreement compared to choosing to default and switching to autarky

$$\Delta_t (q) = \eta_t e^{-\rho t} \frac{1}{\alpha} \left( \left[ \left( \frac{1}{q} \right) \frac{\lambda_A}{\eta_t} e^{\rho t} \xi_B (q) \right]^{-\frac{\alpha}{1-\alpha}} - \delta_{A,t}^\alpha \right)$$

In equilibrium optimal surplus utility of Investor A, denoted $\Delta_t^* (q)$, solves the stopping time problem:

$$\Delta_t^* (q) = \sup_{\tau} \text{E}^B_t \int_t^\tau \eta_u e^{-\rho (u-t)} \frac{1}{\alpha} \left( \left[ \left( \frac{1}{q} \right) \frac{\lambda_A}{\eta_u} e^{\rho u} \xi_B (q) \right]^{-\frac{\alpha}{1-\alpha}} - \delta_{A,u}^\alpha \right) du$$

Moreover, $\Delta_t^* (q)$ can be also expressed as

$$\Delta_t^* (q) = \Delta_t^{unc} (q) + P_t (q)$$

where
\[
\Delta^\text{unc}_t (q) = E_t^B \int_t^{\infty} \frac{\eta_u e^{-\rho (u-t)}}{\eta_t} \left[ \left( \frac{1}{q} \right) \frac{\lambda_A e^{\rho u} \xi_B (q)}{\eta_u} \right]^{-\frac{1}{\alpha}} d\eta_u - \delta^\alpha_{A,u} du
\]

is the surplus discounted expected utility of staying in the risk sharing agreement for ever compared to choosing to default and switching to autarky, under full commitment but with a state price density associated with limited commitment. In turn,

\[
P_t (q) = \sup_{t \in [0,T]} \int_0^{\infty} \frac{\eta_u e^{-\rho (u-t)}}{\eta_t} \left[ \left( \frac{1}{q} \right) \frac{\lambda_A e^{\rho u} \xi_B (q)}{\eta_u} \right]^{-\frac{1}{\alpha}} d\eta_u - \delta^\alpha_{A,u} du
\]

is an American put option written on the utility surplus of staying in the risk sharing agreement compared to switching to autarky for ever under full commitment; with null strike and with payoff \(\max [-\Delta^\text{unc}_t (q), 0]\) upon exercise at time \(t\).

The value function \(J (1 + \beta^*_t)\) of the solvency constrained Investor A can be expressed as the difference between his value function if he ignored the solvency constraints \(J (1)\) and the expected value of the stream of American puts needed in order to enforce the solvency constraints at all times and states:

\[
J (1 + \beta^*_t) = J (1) - \int_0^{\infty} P_t (q) dq
\]

Let \(q^*_t = \inf \{ q : \Delta^*_t (q) = 0 \}\) denote the optimal (stochastic) exercise boundary. Then the equilibrium shadow price of solvency constraints is given by:

\[
\beta^*_t = \sup_{v \in [0,t]} \{ q^*_v \} - 1
\]

and the optimal exercise boundary satisfies:

\[
\Delta^*_t (q^*_t) = \Delta^\text{unc}_t (q^*_t) + P_t (q^*_t) = 0
\]

**Proof**: See the appendix.

Existence of a unique solution to the stopping time problem set up in the proposition is proven in the appendix. This solution involves a stochastic barrier \(q^*_t\) which represents the optimal exercise boundary. Optimal exercise occurs whenever the boundary process reaches the shadow price of solvency constraints: \(q^*_t = 1 + \beta^*_t\). Since by definition \(1 + \beta^*_t\) is non-decreasing, it follows that \(\beta^*_t = \sup_{v \in [0,t]} \{ q^*_v \} - 1\) is the equilibrium shadow price of solvency constraints.

The results on the proposition are intuitive and they follow from the individual optimization of Investor A assuming an exogenous state price density (see the appendix). However, from the class
of state price densities characterized by proposition 2, we are interested particularly in the one which is consistent with both market clearing and optimal consumption, which is given in equation (33). Using the optimal consumption based on that particular state price density in equation (34), we have from Proposition 2 that the optimal exercise boundary satisfies

\[
0 = \Delta_t^{unc}(q_t^*) + P_t(q_t^*)
\]

\[
= B_t \int_{t}^{\infty} -\frac{\eta}{\eta} e^{-(u-t)\frac{1}{\alpha} \delta_{A,u}^{\alpha} \omega^\alpha (\eta_u, q_u^*) (1 + r_u)^\alpha - 1} du
\]

\[
+ \sup_{\tau} E_{t}^{B} \int_{\tau}^{\infty} -\frac{\eta}{\eta} e^{-(u-t)\frac{1}{\alpha} \delta_{A,u}^{\alpha} \omega^\alpha (\eta_u, q_u^*) (1 + r_u)^\alpha - 1} du
\]

Recall that \(P_t(q_t^*)\) is an American put option with payoff \(\max\{ -\Delta^{unc}_t(q_t^*) , 0\}\) upon exercise at time \(t\). It is of particular interest to compute the boundary \(q_t^*\) associated with optimal exercise of the option to default and switch to autarky forever. In order to do so, it is convenient to switch to the early exercise premium (EEP) representation of the American put. Detemple and Tian (2002) have shown that when several state variables influence the optimal exercise of the American put, a single exercise boundary determines the exercise region under very general conditions, which are satisfied in the present case.\(^{10}\) Although in general the full vector of exogenous state variables determines the default boundary of Investor \(A\), I take as a pivotal variable \(\eta_u\) both for tractability and because it makes the intuition clear.\(^{11}\) In particular, assuming that all the other state variables remain constant, the payoff of the American put, \(\max\{ -\Delta^{unc}_t(q_t^*) , 0\}\), is strictly decreasing in \(\eta_u\) for a higher level of \(\eta_u\). Investor \(A\) optimal consumption share of aggregate endowment increases, thus the unconstrained surplus utility \(\Delta^{unc}_t(q_t^*)\) increases and the payoff of the put decreases. Let \(x_t\) be the optimal exercise boundary associated with sentiment risk \(\eta_t\), then immediate exercise is optimal at date \(u\) whenever \(\{\eta_u < x_u\}\). Therefore, the EEP representation of the American put option is:

\[
P_t(q) = B_t \int_{t}^{\infty} -\frac{\eta}{\eta} e^{-(u-t)\frac{1}{\alpha} \delta_{A,u}^{\alpha} \omega^\alpha (\eta_u, q_u^*) (1 + r_u)^\alpha - 1} 1_{\{\eta_u < x_u\}} du
\]

Next, using equations (37) and (38), we get that at the optimal exercise boundary \(\eta_t = x_t\) the following condition must be satisfied:

\[
B_t \int_{t}^{\infty} -\frac{\eta}{\eta} e^{-(u-t)\frac{1}{\alpha} \delta_{A,u}^{\alpha} \omega^\alpha (\eta_u, q_u^*) (1 + r_u)^\alpha - 1} 1_{\{\eta_u > x_u\}} du = 0
\]

The last equation says that, if Investor \(A\) knew with certainty that solvency constraints will never bind in the future, then at the optimal exercise boundary \(\eta_t = x_t\) her discounted expected surplus utility of staying in the risk sharing agreement forever compared to switching to autarky forever would be zero. This means that if Investor \(A\) knew with certainty that solvency constraints will never bind in the future, then the investor’s choice of the optimal time to default and switch to autarky would depend only on comparison of the current utility of staying in the risk sharing agreement for ever versus the current utility of choosing to default and switching to autarky forever. Alternatively, equation (39) can be also written as:

\(^{10}\)I consider the optimal boundary of sentiment \(\eta\). Since it’s drift and volatility are constant, monotonicity with respect to the initial value of sentiment holds. Therefore, a single exercise boundary on \(\eta\) (which depends on \(r\)) determines the optimal default region.

\(^{11}\)I define the boundary of \(\eta\), but the optimal boundary of \(\eta\) will depend on \(r\). The reason for the non-dependence of the boundary on \(\delta A\) is that the utility function is homogeneous, so under a proper change of measure only relative income \(r\) is relevant in the decision to default. This is formalized in Proposition 3.
Utility of switching to autarky forever =

\[ E^B_t \int_t^\infty \frac{\eta_u}{\eta_t} e^{-\rho(u-t)} \left( \frac{1}{\alpha} \delta_{A,u} \right)^\alpha du \]

Utility of staying in the risk sharing agreement forever +

\[ E^B_t \int_t^\infty \frac{\eta_u}{\eta_t} e^{-\rho(u-t)} \left( \frac{1}{\alpha} \delta_{A,u} \right)^\alpha \omega^\alpha (\eta_u, q_u^*) (1 + r_u)^\alpha du \]

\[ + \text{Increase in utility due to the option to default and switch to autarky} \]

The last equation says that, if Investor A knew that there is a positive probability that the solvency constraints may bind in the future, then at the optimal exercise boundary \( \eta_t = x_t \) her discounted expected surplus utility of staying in the risk sharing agreement forever compared to switching to autarky forever plus the value of the option to default and switch to autarky forever in the event that solvency constraints bind in the future is equalized to zero. Therefore, optimal enforcement of solvency constraints occurs only when the surplus utility of staying in the risk sharing agreement forever compared to switching to autarky forever is sufficiently negative.

By using the result in Proposition 1 into equation (39) we obtain that at the boundary \( \eta_t = x_t \) it holds:

\[ 0 = E^B_t \int_t^\infty \frac{\eta_u}{\eta_t} e^{-\rho(u-t)} \left( \frac{1}{\alpha} \delta_{A,u} \right)^\alpha (q^*) \]

\[ \times \left\{ \sum_{j=0}^{\alpha} \left( \begin{array}{c} \alpha \\ j \end{array} \right) [1 - \theta(q^*)]^j \sum_{k=0}^{j} \left( \begin{array}{c} j \\ k \end{array} \right) (1 + r_u)^\alpha - 1 \right\} \eta_u^{\frac{1}{\alpha}} (1 + r_u)^\alpha 1_{\{\eta_u < x_u\}} du \]

Note that in this expression the subscript for \( q^* \) has been removed. This is because the expectation is across the paths were solvency constraints don’t bind (\( \eta_u > x_u \)). Therefore, due to the complementary slackness condition in equation (27) the following events are equivalent

\[ \{\eta_u > x_u\} = \{V_u > 0\} = \{d\beta_u = 0\} = \{dq_u = 0\} \]

Thus, the indicator function inside the expectation ensures that \( q_u^* = q_t^* = q^* \).

At this point, by exploiting the homogeneity in the utility function, a proper change of measure may allow to write the previous expectations independently of \( \delta_{A,u} \), which appears as a multiplicative term.

**Proposition 3.** Denote by \( Z \) a new probability measure defined by

\[ \frac{dQ^Z}{dQ^B} = \left( \frac{(\delta_{A,u})^\alpha}{E^B_t [(\delta_{A,u})^\alpha]} \right) = e^{-\frac{1}{2} (\alpha \sigma_{A})^2 (u-t) + (\alpha \sigma_{A}) (W_{T}^{B} - W_{A}^{T})} \]

such that the Brownian motion \( W^Z = \left( W_{A}^{Z}, W_{B}^{Z} \right) \) is related to the Brownian motion \( W^B = \left( W_{A}^{B}, W_{B}^{B} \right) \) by

\[ dW_{T}^{B} = dW_{A}^{Z} + \alpha \sigma_{A} dt \]

17
\[
dW_{\delta B,t} = dW_{Z,t}
\]

Let \( \tilde{\rho} = \rho - \left[ \alpha \hat{\mu}_B^B + \frac{1}{2} \alpha (\alpha - 1) \sigma_B^2 \right] \) and assume \( \tilde{\rho} > 0 \). Under the new measure \( Z \), equation (41) can be written such that at the optimal exercise boundary \( \eta_t = x_t \) it holds:

\[
0 = \mathbb{E}_t^Z \left[ \int_t^\infty \frac{\eta_u}{\eta_t} e^{-\tilde{\rho}(u-t) - u} \frac{1}{\alpha} \theta^\alpha (q^*) \right. \\
\left. \sum_{j=0}^{-\alpha} \left( \begin{array}{c} -\alpha \\ j \end{array} \right) [1 - \theta (q^*)]^j \sum_{k=0}^j \left( \begin{array}{c} j \\ k \end{array} \right) (-1)^k \eta_u^{\frac{j-k}{\alpha}} \right] (1 + ru)^\alpha - 1 \right] 1_{(\eta_u > x_u)} du
\]

**Proof:** See the appendix.

The last proposition implies that under a proper change of measure, the expected discounted utility of staying in the risk sharing agreement and the expected discounted utility of choosing to default and switching to autarky forever are independent of the realization of \( \delta_{A,t} \). Therefore, under this measure we also expect the optimal exercise boundary to be independent of \( \delta_{A,t} \). Intuitively, due to homogeneity of the utility function only sentiment risk and the relative endowment are relevant state variables in the decision to stay in the risk sharing agreement versus choosing to default and switching to autarky forever.

Taking this into consideration, by inspecting the equation for the optimal exercise boundary in Proposition 3 we note that \( x \) is a function of the ratio of endowment \( r \) and of the shadow price of solvency constraints through \( \Phi (q^*) \). Therefore, we expect the optimal exercise boundary of \( \ln \eta_t \) to be of the form \( \tilde{x}_t = \tilde{x}_t (\theta, \ln r_t) \). Define the inverse of \( \tilde{x}_t (\theta, \ln r_t) \) with respect to \( \theta \) as \( \Phi (\cdot, \ln r_t) \). Since the instantaneous utility of staying in the risk sharing agreement is monotonically increasing in \( \theta \), \( \tilde{x}_t (\theta, \ln r_t) \) must be monotonically decreasing in \( \theta \), and it follows that a unique inverse \( \Phi (\cdot, \ln r_t) \) exists. Because this inverse is decreasing in its argument, the continuation region can be written as:

\[
\{ \tilde{x}_t (\theta, \ln r_t) < \ln \eta_t \} = \{ \theta > \Phi (\ln \eta_t, \ln r_t) \}
\]

In this region the dynamic multiplier \( \beta_t \) stays flat, it increases at times when immediate exercise is optimal. Therefore, we have that the optimal \( \theta (1 + \beta^*_t) \) satisfies:

\[
\theta (1 + \beta^*_t) = \sup_{\nu \in [0,t]} \Phi (\ln \eta_t, \ln r_t)
\]

It follows that \( \beta_t \) can be computed from inverting \( \theta (1 + \beta^*_t) \) using its definition in Proposition 1. This leads to the equilibrium shadow price of solvency constraints, as summarized in the next proposition.

**Proposition 4.** The equilibrium shadow price of solvency constraints which precludes default and switching to autarky at all times and states is given by

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12 Indeed the optimal solution is independent of the realization of \( \delta_{A,t} \), however the parameters governing the process (\( \hat{\mu}_B^B \) and \( \sigma_B^B \)) do influence the solution.

13 The optimal exercise boundary is independent of time due to the infinite horizon assumed.

14 Because \( r_t \) and \( \eta_t \) are distributed lognormal it will be convenient for some calculations to have them expressed in logs, therefore the notation may be changing between logs and levels for these variables.
where $\Phi(\cdot, \ln r_t)$ is the inverse of $\hat{x}_t(\theta, \ln r_t)$ with respect to $\theta$, and $\hat{x}_t(\theta, \ln r_t)$ is the optimal exercise boundary of $\ln \eta_t$ such that continuation in the risk sharing agreement is optimal if $\ln \eta_t > \hat{x}_t(\theta, \ln r_t)$. The optimal exercise boundary $\hat{x}_t(\theta, \ln r_t)$ satisfies:

$$
0 = \frac{e^{\beta_t}}{\eta_t} \int_t^{\infty} e^{-\hat{\rho}_u} \frac{1}{\alpha} \left( \theta^\alpha \sum_{j=0}^{-\alpha} \binom{-\alpha}{j} \frac{1 - \theta^j}{j} \sum_{k=0}^j \binom{j}{k} (-1)^k \right.

\times H \left( \hat{x}_t(\theta, \ln r_t), r_t, u - t, \hat{x}_t(\theta, \cdot), 1 + \frac{k - j}{1 - \alpha}, \alpha \right) \left. \right) - H \left( x(\theta, r_t), r_t, u - t, \hat{x}_t(\theta, \cdot), 1, 0 \right) \, du
$$

where $H(\eta_t, r_t, u-t, \hat{x}_t(\theta, \cdot), \chi, \epsilon) = E^P_u [\eta_u(1+r_u)^\epsilon 1_{\{\eta_u > x_u\}}]$ is given by

$$
H(\eta_t, r_t, u-t, \hat{x}_t(\theta, \cdot), \chi, \epsilon) = \int_{-\infty}^{+\infty} e^{\bar{\pi}(\ln \eta_t, \ln r_t, \ln r_u, \chi) + \frac{1}{2} \pi^2(\chi) u - t) \frac{\ln r_u - \mu_r (\ln r_t, u - t)}{\sigma_r (u - t)} N \left( d(\ln \eta_t, \ln r_t, \ln r_u, \hat{x}_t(\theta, \ln r_t), \chi, u - t) \right) \left( 1 + e^{\ln r_u} \right)^\epsilon d(\ln r_u)
$$

and where $N(\cdot)$ and $n(\cdot)$ are the standard normal cumulative distribution function and the standard normal density function, respectively. The functions $\bar{\pi}(\cdot), \pi(\cdot)$ and $d(\cdot)$ appearing in the expression for $H(\cdot)$ are given in the appendix.

The initial value is $\Phi(\eta_0, r_0) = \frac{(\frac{\lambda_B}{\lambda_A})^{\frac{1}{1-\alpha}}}{1 + (\frac{\lambda_B}{\lambda_A})^{\frac{1}{1-\alpha}}}$ such that $\beta_0 = 0$ and solvency constraints are not binding in the initial period.

Proof: See the appendix.

The next proposition shows that the binding of solvency constraints is persistent.

**Proposition 5.** Under the objective probability measure, which governs empirical realizations, the probability of solvency constraints binding in any future period $u$ is decreasing in the current level of sentiment $\eta_t$ for $u > t$.

Proof: See the appendix.
and periods of non-binding of solvency constraints tend to be followed by periods of non-binding of solvency constraints.

Intuitively, solvency risk is persistent because it is driven by the shadow price of solvency constraints, which is related to the optimal exercise boundary of a sequence of American contingent claims in the excess utility of staying in the risk sharing agreement compared to choosing to default and switching to autarky forever. When the solvency constraint binds, the shadow price of solvency constraints increases in order to preclude default and switching to autarky. This increase is just enough as to generate indifference between staying in the risk sharing agreement and choosing to default and switching to autarky, hence it leaves the level of sentiment close to the exercise boundary of the next American put, with a higher probability of binding repeatedly, or persistence in the binding of solvency constraints.

5 Solution Methodology

In order to fully characterize the equilibrium we need to compute the optimal exercise boundary $\hat{x}_t(\theta, \ln r_t)$. Starting from Proposition 4, it can be computed numerically. In particular, I adapt the methods developed by Detemple and Tian (2002) and Ibáñez and Zapatero (2004) for the current problem. The main advantage compared to their procedure, is that by having an infinite horizon I don’t need to implement the solution backwards. Instead I just iterate for the points in the grid until convergence.

Denote by $\tilde{H}$ an approximation for the function $H(\eta_t, r_t, u - t, x(\theta, \cdot), \chi, \epsilon)$ defined in Proposition 4

$$
\tilde{H}(\eta_t, r_t, u - t, x(\cdot), \chi, \epsilon) = \int_{\ln r}^{\ln r + \Delta \ln r} e^{r_t} \rho_t \ln r_t, \ln r, \chi, u - t) + \frac{1}{2} \sigma^2(\chi, u - t) \eta_t \left[ \frac{\ln r_t - \mu_t (\ln r_t, u - t)}{\sigma_r (u - t)} \right]
$$

$$
\times N \left[ d (\ln r_t, \ln r, \ln r, \tilde{x}_t(\theta, \ln r_t), \chi, u - t) \right] \left( 1 + e^{\ln r} \right) \epsilon d (\ln r)
$$

where $\tilde{x}_t(\theta, \ln r_t)$ is the optimal exercise boundary for $\ln \eta_t$ and $[\ln r, \ln r]$ is the range of $\ln r$.

The optimal exercise boundary can be computed as follows. Fix the level of $\theta$ to $\theta = \bar{\theta}$, and notice it can be taken as a given constant in the solution procedure because of the indicator function present inside the expectation in the definition of the function $H(\cdot)$ in proposition 42. Divide the range $[\ln r, \ln r]$ into $M$ equal subintervals and let $\Delta \ln r = \frac{\ln r - \ln r}{M}$. For each $i = 0$, $\ldots$, $M$ set $\ln r_i = \ln r + i \Delta \ln r$. Then, in each iteration $n$ we solve at each point in the grid $p = 0$, $\ldots$, $M$ for $x(\ln r_p)$ from the following equation:

$$
0 = \frac{e^{\tilde{p}_t}}{\eta_t} \int_t^{\infty} e^{-\rho u} \frac{1}{\alpha} \left( \theta^\alpha \sum_{j=0}^\infty \left( \frac{-\alpha}{j} \right) (1 - \theta)^j \sum_{k=0}^j \left( \frac{j}{k} \right) (-1)^{j-k} \left[ H \left( \tilde{x}_n(\theta, \ln r_p), r_p, u - t, \tilde{x}_{n-1}(\theta, \ln r_s), 1 + \frac{k - j}{1 - \alpha}, \alpha \right) \right] \Delta \ln r \right) \Delta \ln r
$$

$$
- \sum_{s=0}^M \left[ H \left( \tilde{x}_n(\theta, \ln r_p), r_p, u - t, \tilde{x}_{n-1}(\theta, \ln r_s), 1, 0 \right) \Delta \ln r \right] du
$$
In the first iteration we assume a naive guess for 
\( x_0 (\theta, \ln r_s) = \frac{-1}{\varepsilon} \) for all \( s \), with \( \varepsilon \) been a very small number, such that this approximates the case when the optimal exercise boundary tends to minus infinity. Starting the second iteration we use equation (45) until convergence is reached and we find a stationary grid such that 
\[ x_{n+1} (\theta, \ln r_p) \rightarrow x_{n+1} (\theta, \ln r_p) \] .

This method produces an approximate optimal exercise boundary \( \hat{x} (\theta, \ln r_p) \) for a given level of \( \theta = \bar{\theta} \). By performing the same procedure for different levels of \( \theta \) in a grid between zero and one, we obtain an approximate optimal exercise boundary surface: \( \hat{x} (\theta, \ln r_p) \). Inverting this function with respect to \( \theta \) allows to get an approximate function \( \Phi (\ln \eta_t, \ln r_t) \), which leads to computation of the optimal shadow price of solvency constraints \( \beta_t \), as specified in Proposition 4.

The left panel of Figure 1 shows the approximate function \( \hat{x} (\theta, \ln r_t) \). In approximating \( \hat{x} (\theta, \ln r_t) \) I constructed a grid of 30 points for \( \ln r_t \) with values of \( r_t \) between 0.01 and 5, and a grid of 10 points for \( \theta \) with values between 0.05 and 0.95. Convergence was readily achieved in the second iteration, up to two decimals precision. The optimal exercise boundary is decreasing in both arguments, which is intuitive because higher levels of both \( \theta \) and \( r_t \) increase the surplus utility of continuation in the risk sharing agreement compared to choosing to default and switching to autarky forever.

The right panel of Figure 1 shows the approximate function \( \Phi (\ln \eta_t, \ln r_t) \) which is the inverse of \( \hat{x} (\theta, \ln r_t) \) with respect to \( \theta \). In order to approximate \( \Phi (\ln \eta_t, \ln r_t) \) I first interpolated the approximate function \( \hat{x} (\theta, \ln r_t) \). Then, I evaluated each of the 300 grid points for \( x_t \) obtained previously in each of the 30 grid points for \( \ln r_t \), solving numerically for the value of \( \theta \) satisfying \( x_t = \hat{x} (\theta, \ln r_t) \). As expected, the function \( \Phi (\ln \eta_t, \ln r_t) \) is decreasing in both arguments.

6 Equilibrium Financial Securities Pricing

6.1 Interest rate and market prices of risk

Plugging the equilibrium shadow price of solvency constraints in Proposition 4 into the market clearing state price density in equation (33) we get the equilibrium state price density, which in addition to being related to the exogenous state variables, depends also on the full record of events in which the solvency constraints have been binding. The next proposition presents the equilibrium interest rate and market price of risk.

**Proposition 6.** In equilibrium, the return of the riskless bank deposit with value \( B_t \) is given by

\[
\frac{dB_t}{B_t} = \int_i dt - dQ_t
\]

with

\[15\] The parameters used in the numerical procedures and simulations in all the figures presented in the paper are given in Table 1.
\[ i_t = \rho + (1 - \alpha) \left( \frac{1}{1 + r_t \delta_A} + \frac{r_t}{1 + r_t \delta_B} \right) - \frac{1}{2} (1 - \alpha) (2 - \alpha) \left[ \left( \frac{1}{1 + r_t} \right)^2 \sigma_{\delta_A}^2 + \left( \frac{r_t}{1 + r_t} \right)^2 \sigma_{\delta_B}^2 \right] \\
- \frac{1}{2} \left( \frac{\alpha}{1 - \alpha} \right) \left( \mu_{\delta_A}^2 + \mu_{\delta_B}^2 \right) \omega (\eta_t, 1 + \beta_t) [1 - \omega (\eta_t, 1 + \beta_t)] \\
- (1 - \alpha) \omega (\eta_t, 1 + \beta_t) \left( \frac{1}{1 + r_t} \sigma_{\delta_A} \mu_{\delta_A} + \frac{r_t}{1 + r_t} \sigma_{\delta_B} \mu_{\delta_B} \right) \]

and

\[ dQ_t = \omega (\eta_t, 1 + \beta_t) \frac{d\beta_t}{1 + \beta_t} \]

Similarly, the equilibrium market price of risk is given by:

\[ \kappa_t^B = \begin{bmatrix} \kappa_{\delta_A,t}^B \\ \kappa_{\delta_B,t}^B \end{bmatrix} = (1 - \alpha) \frac{1}{1 + r_t} \begin{bmatrix} \sigma_{\delta_A} \mu_{\delta_A} \\ r_t \sigma_{\delta_B} \mu_{\delta_B} \end{bmatrix} + \omega (\eta_t, 1 + \beta_t) \frac{\mu_{\delta_A}}{\mu_{\delta_B}} \]

Proof: It follows from applying Itô Lemma to equation (33) and identifying terms with equation (20).

In equilibrium, the return of the riskless bank deposit (interest rate) has two components. The first is a locally riskless interest rate which has the familiar structure in the case of an exchange economy with heterogeneous beliefs (see Dumas, Kurshev and Uppal (2005) for a detailed analysis on this component). Unlike models with exogenous portfolio constraints or incomplete markets, the return of the riskless bank deposit also has a singular component, which is tied to the occurrence of a binding solvency constraint. \(^\text{16}\) At times when the solvency constraint binds, \(C_{\delta,t}^A\) experiences a permanent increase. In such scenarios, the interest rate drops just enough to reduce aggregate consumption immediately. Due to no arbitrage, the expected stock returns must also include the same singular component \(Q_t\). Therefore, when solvency constraints bind expected stock returns are decreased.

### 6.2 Stock Return Volatility

The last subsection described the interest rate and obtained the instantaneous market price of risk. In this subsection I obtain the equilibrium stock return volatility and show the role of the market price of risk as a common component of short-run stock return volatility.

**Proposition 7.** The diffusion vectors of the stock-market return are given by:

\[ \sigma_{S,t} = \begin{bmatrix} \sigma_{S,A,t} \\ \sigma_{S,B,t} \end{bmatrix} = \sigma_{S,A}^* + \Sigma_{S,A}, \quad \sigma_{S} = \begin{bmatrix} \sigma_{S,A}^* \\ \sigma_{S,B}^* \end{bmatrix} = \sigma_{S,A}^* + \Sigma_{S,B} \]

where \( \sigma_{S_i}^* \) represents the diffusion vector of stock \( S_i \) associated with short run volatility, and \( \Sigma_{S_i} \) represents the diffusion vector of stock \( S_i \) associated with long run volatility. They are given by:

\(^{16}\) Singular components in the interest rate also arise when marginal utility of agents is bounded at zero [see Karatzas, Lehockzy, Shreve and Xu (1991)], and when liquidity constraints are imposed in general equilibrium [see Detemple and Serrat (2003)].
the fundamental asset pricing equations and the long-run volatilities are given by
\[
\sigma^*_{SA_t} = \sigma_{\delta_A} + \kappa^B_t, \quad \sigma^*_{SB_t} = \sigma_{\delta_B} + \kappa^B_t
\]
\[
\Sigma_{SA} = \int_{t}^{\infty} \mathbb{E}_t^B \left[ \delta_{A,u} \mathcal{D}_t \left( \xi^B_u \right) \right] du, \quad \Sigma_{SB} = \int_{t}^{\infty} \mathbb{E}_t^B \left[ r_u \delta_{A,u} \mathcal{D}_t \left( \xi^B_u \right) \right] du
\]
where \( \mathcal{D}_t \) is the Malliavin derivative operator, and \( \mathcal{D}_t \left( \xi^B_u \right) \) is given by
\[
\mathcal{D}_t \left( \xi^B_u \right) = -\xi^B_u \left[ \frac{\hat{\kappa}_u - \omega \left( \eta_u, \beta_u \right)}{1 + \beta_u} \right]
\]
with
\[
\frac{\mathcal{D}_t \left( \beta_u \right)}{1 + \beta_u} = \frac{1}{1 - \alpha} 1_{\{v(\theta, u) \geq t\}} \frac{\frac{\partial \Phi}{\partial \tau} \left( \eta_t \left( \xi^B_u \right), \rho \left( \xi^B_u \right) \right)}{\Phi \left( \eta_t \left( \xi^B_u \right), \rho \left( \xi^B_u \right) \right)} \left[ \frac{\eta_t \left( \xi^B_u \right)}{1 - \Phi \left( \eta_t \left( \xi^B_u \right), \rho \left( \xi^B_u \right) \right)} \right]
\]
and \( v \left[ \tau, u \right] \) is the last time in \( \left[ \tau, u \right] \) at which the solvency constraint bound:
\[
v \left[ \tau, u \right] = \sup \left\{ v \in \left[ \tau, u \right] : \Phi \left( v \right) = \sup_{s \in \left[ \tau, u \right]} \Phi \left( s \right) \right\}
\]

The total volatilities are defined as
\[
\sigma_{SA, t}^{\text{total}} = \sqrt{\sigma^*_{SA, t}^2 + \sigma_{SA, t}^2}, \quad \sigma_{SB, t}^{\text{total}} = \sqrt{\sigma^*_{SB, t}^2 + \sigma_{SB, t}^2}
\]
the short-run volatilities are defined as
\[
\sigma_{SA, t}^{\text{short}} = \sqrt{\sigma^*_{SA, t}^2 - \sigma_{SA, t}^2}, \quad \sigma_{SB, t}^{\text{short}} = \sqrt{\sigma^*_{SB, t}^2 - \sigma_{SB, t}^2}
\]
and the long-run volatilities are given by
\[
\sigma_{SA, t}^{\text{long}} = \sqrt{\sigma_{SA, t}^2 - \sigma^*_{SA, t}^2}, \quad \sigma_{SB, t}^{\text{long}} = \sqrt{\sigma_{SB, t}^2 - \sigma^*_{SB, t}^2}
\]
By construction, all of them are strictly positive and satisfy
\[
\left( \sigma_{SA, t}^{\text{total}} \right)^2 = \left( \sigma_{SA, t}^{\text{short}} \right)^2 + \left( \sigma_{SA, t}^{\text{long}} \right)^2
\]
\[
\left( \sigma_{SB, t}^{\text{total}} \right)^2 = \left( \sigma_{SB, t}^{\text{short}} \right)^2 + \left( \sigma_{SB, t}^{\text{long}} \right)^2
\]

Proof: See the appendix.

The volatility decomposition in Proposition 7 has an economic interpretation that follows from the fundamental asset pricing equations
\[
S_{A,t} = \int_{t}^{\infty} \mathbb{E}_t^B \left[ \xi^B_u \delta_{A,u} \right] du \quad \text{(46)}
\]
\[
S_{B,t} = \int_{t}^{\infty} \mathbb{E}_t^B \left[ r_u \delta_{A,u} \right] du \quad \text{(47)}
\]
These equations relate the prices of financial securities to three variables: current state prices \( \xi_t \), future state prices \( \xi_{t+1} \), and future cashflows represented by the future delivery of promised endowments \( \delta_{t, u} \). The equilibrium stock return volatility depends on the sensitivity of stock returns to the current shocks in the economy through each of these variables. The first term in the short run diffusion vector of volatility \( \sigma_{S_t}^B \) represents the sensitivity to current shocks through their effect on future cashflows. The second (common) term \( \kappa_t^B \) is the sensitivity to current shocks through their effect on the current state price density. Finally, the long run diffusion vector of volatility \( \Sigma_{S_t} \) represents the sensitivity to current shocks through their effect on the future state price density.

It is worth noting that the short-run volatility is represented by transitory sensitivity to current shocks, which have an effect in the stock price only through current state prices and cashflows.\(^\text{17}\) In contrast, long-run volatility is related to a more permanent sensitivity to current shocks, including their effect in the future evolution of state prices.

### 6.2.1 Short-run Stock Return Volatility

In order to provide further insights on short-run stock return volatility, \( \sigma_{S_t}^{\alpha_t} \), notice that the market price of risk can be broken into three components, each associated with a different risk factor

\[
\kappa_t^B = (1 - \alpha) \frac{1}{1 + r_t} \left[ \sigma_{\delta_A} \frac{\sigma_{\delta_B}}{\eta (1 - \delta_B)} \right] + \omega (\eta_t, 1) \left[ \frac{\mu_{\delta_A}}{\mu_{\delta_B}} \right] + \left[ \omega (\eta_t, 1 + \beta_t) - \omega (\eta_t, 1) \right] \left[ \frac{\mu_{\delta_A}}{\mu_{\delta_B}} \right] \tag{48}
\]

where
- endowment risk
- sentiment risk
- solvency risk

The first component is associated with endowment risk, which is time varying due to the "two-trees" feature of the model. Moreover, since the shares of aggregate endowment \( \frac{1}{1 + r_t} \left( \frac{1}{1 + r_t} \right) \) fluctuate stochastically between zero and one, we expect this component to be persistent. The second component is associated with sentiment risk, and would be present even in the absence of solvency constraints. Because the optimal share of aggregate consumption in the absence of solvency constraints \( \omega (\eta_t, 1) \) fluctuates stochastically between zero and one, we also expect this component to be persistent. These two components already reproduce some level of persistence in volatility or GARCH effects. The third component is associated with the risk of the binding of solvency constraints, which is denoted solvency risk. This component is identified for the first time in this paper, and this is the main contribution. Remarkably, this component provides consistency of the model with several empirical facts on stock return volatility, as will be documented next.

The solvency risk component is clearly related to sentiment risk, because it is the risk of scaling up sentiment if solvency constraints bind. Thus, it is the interaction of solvency constraints with sentiment risk that generate solvency risk. Therefore, this component would not be present in the market price of risk in an economy without sentiment risk, even if endogenous solvency constraints were imposed. For instance, this component is absent in models with endogenous liquidity constraints but without sentiment risk as in Detemple and Serrat (2003). It is also absent in models with endogenous solvency constraints but without sentiment risk as in Alvarez and Jermann (2000, 2001). Similarly, this component would not be present in the market price of risk in an economy without endogenous solvency constraints, even if heterogeneity of beliefs were assumed. For instance, this component is absent in models of heterogenous beliefs under full commitment.

\(^{17}\)Since dividends are geometric Brownian motions, the effect of current shocks on stock prices through future cash flows is equivalent to the effect of current shocks on stock prices through current cash flows: \( \sigma_{\delta_A} \).

24
Having characterized the market prices of risk and its dynamics, now we turn into a deeper analysis of short-run stock return volatility, both conditionally and unconditionally.

A. Conditional short-run stock return volatility

Figure 2 plots the conditional short-run stock return volatility $\sigma_{S,t}^{sr}$ as a function of the ratio of endowments $r_t$, sentiment $\eta_t$ and solvency risk $\beta_t$, respectively.\(^{18}\) In the three plots the dashed line is for the case of full agreement ($\overline{p}_{\delta_A} = \overline{p}_{\delta_B} = 0$) and the solid line is for the case of disagreement across investors ($\overline{p}_{\delta_A} = \overline{p}_{\delta_B} = 0.25$). The first plot shows that short-run stock return volatility is influenced by the distribution of endowments. When this ratio gets closer to one such that endowments become identical across investors, then benefits of risk sharing through trade in the financial markets is reduced, which generates a decrease in the volatility of stock return. This endowment risk component is present independent of the level of disagreement, although the overall stock return volatility would increase if disagreement were higher, due to sentiment and solvency risks.

In the bottom two plots, it is shown that both sentiment and solvency risk increase short-run stock return volatility, but this occurs only if investors disagree. Therefore, disagreement is crucial in order to get the effects of sentiment and solvency risk into short-run stock return volatility.

B. Unconditional short-run stock return volatility

Figure 3 shows the unconditional short-run stock return volatility $\sigma_{S,t}^{sr}$, as well as the unconditional dynamics of the relevant state variables, obtained from simulating one path of the model across 46 years (11583 daily observations) of data. In the first row are plotted the ratio of endowments and the optimal consumption share of aggregate endowment corresponding to investor $A$. In the latter case, the solid line represents the case without solvency risk ($\beta_t = 0$ for all $t$) and the dashed lines is the case with solvency risk ($\beta_t$ unrestricted). For instance, the plot shows that solvency risk prevents sentiment from vanishing, since it scales it up when it reduces too much in order to preclude default.

The second row of Figure 3 plots the endogenous state variables, which are governed by solvency risk. The figure makes clear that the dynamics of solvency risk have a very different nature compared to those of exogenous state variables. Endowment risk and sentiment risk are driven by exogenous state variables, which are continuous stochastic processes represented by geometric Brownian motions. In contrast, the solvency risk component is driven by the interaction of an exogenous state variable and an endogenous state variable. In particular, it will have different dynamics depending on solvency constraints binding or not. Because the binding of solvency constraints is persistent (see proposition 5), this component also clusters but at a different frequency compared to the clustering of the endowment and sentiment risk components.

The left plot of the third row presents three curves in it: the solid curve is for the case of heterogeneous beliefs and limited commitment ($\beta_t$ and $\eta_t$ unrestricted); the dashed curve is for the case of heterogeneous beliefs and full commitment ($\beta_t = 0$ for all $t$ and $\eta_t$ unrestricted) and the

\(^{18}\)Since results are very similar for both stocks, for simplicity we take stock $A$ for all the empirical illustrations.
dotted curve is for the case of homogeneous beliefs and full commitment ($\beta_t = 0$ and $\eta_t = 1$ for all $t$). In the case of two-trees without sentiment and solvency risks, the persistence in volatility is very low, because the volatility of dividends is too small compared to that of stock return, as pointed out by Shiller (1981). Therefore, the fluctuation between zero and one in the shares of aggregate endowment $\left( \frac{1}{1+r}, \frac{r}{1+r} \right)^T$ is small. When sentiment is added, the persistence of volatility increases, because the volatility of sentiment (disagreement) is much higher than the volatility of endowments. Thus, the fluctuation in the optimal shares of aggregate consumption in the absence of solvency constraints $\omega(\eta_t, 1)$ is considerably higher. Finally, when the full model with sentiment and solvency risk is considered the volatility persistence is further increased, due to the persistence in the binding of solvency constraints.

The remaining three plots show, for the same simulation of shocks, respectively, the difference between: (i) returns obtained assuming "two trees" minus returns obtained using only "one tree", (ii) returns obtained assuming "two trees and sentiment" minus returns using only "two trees", and (iii) returns obtained assuming "two trees, sentiment and solvency risk" minus returns obtained using only "two trees and sentiment". These plots show the marginal contribution to volatility clustering from each risk factor for a particular path of shocks. It is interesting to note that volatility clustering generated by sentiment and solvency risks present different persistence and occur at different frequencies. In particular, the clustering of volatility generated by solvency risk is very much in line with the dynamics of $\beta$ and the singular component $Q$, in the second row of Figure 3.

Related to the last point, by comparing the right plots of the second and fourth rows in Figure 3, it becomes clear that if the population facing limited commitment is relatively pessimistic (as assumed in the simulations, see Table 1) then in periods of binding solvency constraints there is a negative relation between stock return volatility and stock returns. In that case, when solvency constraints bind the market price of solvency risk increases and, correspondingly, short-run stock return volatility increases. In contrast, if the population facing limited commitment was relatively optimistic, in the event of solvency constraints binding the market price of solvency risk would decrease and, correspondingly, short-run stock return volatility would decrease. Regardless on which investor is relatively optimistic, the stock return decreases following the binding of solvency constraints, due to the increment of the singular factor $Q_t$. Therefore, in the current model the relation between stock return volatility and stock return depends on solvency constrained investors been relatively optimistic or pessimistic within the sample period. This may help explain the conflicting empirical evidence in the relation between stock return volatility and stock returns.

C. Simulation results

The last subsection described the unconditional short-run stock return volatility dynamics based on the simulation of one path of the model. Now I conduct a Montecarlo simulation experiment to investigate the ability of the model to reproduce the empirical regularities in short-run stock return volatility. First, I simulate one sample path of 11,583 daily observations (equivalent to 46 years, which is the length of data available for S&P500) of the shocks $\left( W^B_{\delta_A}, W^B_{\delta_B} \right)$. Then, using the parameters in Table 1, I simulate forward the markovian system of exogenous state variables $\{\delta_A, r, \eta\}$ in equations (3), (12) and (11), by using an Euler-Maruyama discretization scheme. Next, using propositions 6 and 7, the numerical approximation of the equilibrium shadow price of solvency constraints, as well as equations (16) and (21), I reconstruct the short-run returns according to three possible scenarios:
1. **Two-trees**: this scenario corresponds to a simple two-trees economy without sentiment and solvency risks. It has been established that the two trees feature generates stochastic volatility as the distribution of income varies through time. So I analyze if this effect alone could reproduce volatility clustering.

2. **Two-trees and sentiment risk**: this scenario adds sentiment risk to the previous one, but omits solvency risk. The aim is to investigate if adding sentiment would provide a better fit of volatility compared with the two trees scenario.

3. **Two-trees, sentiment and solvency risk**: this is the complete model developed in this paper, featuring two-trees endowments, sentiment and solvency risks. We compare the volatility patterns reproduced by this full model with respect to that of the two previous ones.

Then, for the short-run return series simulated in each of these scenarios, I compute the estimated parameters and heteroskedasticity consistent standard errors of an EGARCH(1,1) model, using a typical EGARCH formulation:\(^{19}\)

\[
R_t = c + \sigma_t \varepsilon_t, \quad \varepsilon_t \sim N(0,1) \\
\ln \sigma_t^2 = c_0 + c_1 \frac{\varepsilon_{t-1}}{\sigma_t} + c_2 \frac{\varepsilon_{t-1}}{\sigma_t} + c_3 \ln \sigma_{t-1}^2
\]  

The same procedure is repeated for 10,000 simulated sample paths, and for each scenario the median values for the estimated parameters and p-values associated with the obtained standard errors are reported.\(^{20}\) For comparison purposes, the first column of Table 2 reports EGARCH estimates from the historical S&P500 Composite Index daily returns from January 1st 1964 until May 23rd 2008, obtained from Datastream.\(^{21}\) As it is well known from the empirical literature in volatility, we confirm that the stock returns exhibit significant ARCH and GARCH parameters, as well as a significant leverage effect.

After reporting historical data results, from the second to the fourth column of Table 2 the results of the Montecarlo experiment in the three scenarios described are presented. In particular, the degree and significance of the model in each scenario to capture volatility clustering is analyzed. The second column of Table 2 shows that the **Two Trees** scenario fails to capture any of the features of historical stock return volatility reported in the first column. This is not the case for the **Two-trees and sentiment risk** scenario. At 95% confidence level, the ARCH and GARCH terms are both significant, and at 99% confidence level only the GARCH component remains significant.\(^{22}\)

\(^{19}\)The term "heteroskedasticity consistent standard errors" refers to the fact that we compute the quasi-maximum likelihood (QML) covariances and standard errors using the methods described by Bollerslev and Wooldridge (1992). When the assumption of conditional normality does not hold, the EGARCH parameter estimates will still be consistent, provided the mean and variance functions are correctly specified. The estimates of the covariance matrix and standard errors will not be consistent unless this method is used. In the rest of the paper we refer to them simply as "standard errors".

\(^{20}\)The median is chosen because it is a robust measure of the center of the distribution that is less sensitive to outliers than the mean.

\(^{21}\)For this experiment I choose the simple return instead of the total return (including dividends). The reason is that both series are almost identical, but simple returns are available for a much larger period of time.

\(^{22}\)By "significant" it is meant that for the given confidence level the null hypothesis of the estimated coefficient being equal to zero is rejected.
Therefore, by considering the case of a two trees model with sentiment, some degree of volatility clustering can be obtained, although leverage effects cannot be reproduced. Finally, the Two-trees, sentiment and solvency risk scenario is able to reproduce both volatility clustering and leverage effects. At 95% confidence level, the ARCH term, the GARCH term and the leverage effect term are significant, and at 99% confidence level only the ARCH and GARCH terms remain significant. Thus, we conclude that the model presented in this paper, consisting of a two trees economy with sentiment and solvency risk is able to reproduce volatility clustering as well as leverage effects. Moreover, leverage effects can be only obtained by the interaction of sentiment and solvency risks.

6.2.2 Long-run Stock Return Volatility

Regarding the long-run stock return volatility component $\Sigma_{S_t}$, Proposition 7 shows that it is affected by solvency constraints at two levels. First, the size of the shadow price of solvency constraints affects the sensitivity of the state prices to fluctuations in endowments or sentiment. This effect is explicit in the first term of the Malliavin derivative of state prices $D_t \left( \xi^B_{0,t} \right)$, as well as the denominator of $\Sigma_{S_t}$, both depending on $\beta$. Second, the shadow price of solvency constraints is itself influenced by innovations to the ratio of endowments $r$ and sentiment $\eta$, which generates a new source of long-term volatility. Since $\frac{\partial \Phi}{\partial \eta} < 0$, when Investor $A$ is relatively pessimistic ($\overline{\eta}_t \beta_t > 0$) the effect through sentiment is positive: the possibility of solvency constraints binding in the future increases the long-run stock return volatility through sentiment. Conversely, when Investor $A$ is relatively optimistic ($\overline{\eta}_t \beta_t < 0$) the effect through sentiment is negative: the possibility of solvency constraints binding in the future decreases the long-run stock return volatility through sentiment. Similarly, because $\frac{\partial \Phi}{\partial r} < 0$, the long-run stock return sensitivity to shocks in the endowment of Investor $A$ are positive, but its sensitivity to shocks in the endowment of Investor $B$ are negative. In essence, movements of the exogenous state variables in a direction that increases the likelihood of future binding of solvency constraints increase the long-run stock return volatility.

Because the function $\Phi$ was approximated numerically, its derivatives with respect to its arguments may not be obtained with sufficient precision as to evaluate the long-run stock return volatility by means of Monte Carlo simulations. However, from the analysis presented it is clear that solvency risk has an important contribution also into the long-term stock return volatility. While the empirical experiments in the present paper are based on short-run stock return volatility, empirical work exploring the dynamics of long-run stock return volatility may be a promising avenue for future research.

6.3 Expected Stock Return

Until now I have solved for the instantaneous interest rate, the singular component $Q_t$, the market price of risk and the volatility of stock returns. In order to complete the characterization of equilibrium, in this subsection I solve for the stock expected returns.

**Proposition 8.** A consumption CAPM holds for each stock, which is expressed as

\[ \hat{\mu}^B_{S_{A,t}} - i_t = (1 - \alpha) \text{Cov} \left( \frac{dS_{A,t}}{S_{A,t}}, \frac{d\delta_t}{\delta_t} \right) - \omega (\eta_t, 1 + \beta_t) \text{Cov} \left( \frac{dS_{A,t}}{S_{A,t}}, \frac{d\eta_t}{\eta_t} \right) \]

\[ \hat{\mu}^B_{S_{B,t}} - i_t = (1 - \alpha) \text{Cov} \left( \frac{dS_{B,t}}{S_{B,t}}, \frac{d\delta_t}{\delta_t} \right) - \omega (\eta_t, 1 + \beta_t) \text{Cov} \left( \frac{dS_{B,t}}{S_{B,t}}, \frac{d\eta_t}{\eta_t} \right) \]

where $\delta_t = \delta_{A,t}(1 + r_t)$ is the aggregate endowment.
Proof: It follows from plugging the equilibrium market price of risk found in Proposition 6 into equation (21) and identifying covariance terms.

As in the standard CAPM [Breeden (1979); Duffie and Zame, (1989)], a risky security’s risk premium is positively related to the covariance of its return with aggregate consumption growth (which in equilibrium is equal to the aggregate endowment growth). In the current setup an additional term drives the risk premium. The second term is standard in models with heterogeneous beliefs. As has been pointed out by Basak (2005) this additional term may, under certain conditions, reconcile better with the data concerning the equity premium puzzle [Mehra and Prescott (1985)]. This also holds true in this case and, in addition, because solvency constraints unambiguously decrease the equilibrium interest rate when they bind, the current model may also shed light in the risk-free rate puzzle [Weil (1989)].

However, the main point of this section is the fact that the loading on the sentiment risk factor is scaled up every time solvency constraints bind. In order to see it, note that the sentiment risk factor can be broken as

\[ -\omega(\eta_t, 1 + \beta_t) \text{Cov} \left( \frac{dS_{t,t}}{S_{t,t}}, \frac{d\eta_t}{\eta_t} \right) = -[\omega(\eta_t, 1) + \Delta \omega(\eta_t, 1 + \beta_t)] \text{Cov} \left( \frac{dS_{t,t}}{S_{t,t}}, \frac{d\eta_t}{\eta_t} \right) \]

where

\[ \Delta \omega(\eta_t, 1 + \beta_t) = \omega(\eta_t, 1 + \beta_t) - \omega(\eta_t, 1) > 0 \]

is the strictly positive increment in the pricing of sentiment risk due to solvency constraints.

In a recent contribution, Adrian and Rosenberg (2008) estimate a statistical model for expected stock returns, where the two additional factors to standard CAPM are volatility factors with different persistence extracted from a multifactor GARCH type of model. Moreover, they take the obtained factor loadings and test the ability of such a model in explaining the cross-section of expected returns. They get that the three-factor volatility model compares favorably with four alternative benchmarks, and with a fit that is comparable to the Fama-French model.

Because their volatility model is entirely statistical they lack an interpretation for the volatility components. Therefore they devote a full subsection to trying possible interpretations of their volatility factors. In particular, they relate the less persistent volatility component with the tightness of financial constraints, (which they proxy with market skewness) and the more persistent volatility component with the business cycle (which they proxy with industrial production growth). Although they show empirical evidence on their hypothesis, their paper lacks the foundations on why a CAPM model augmented with two volatility components with different persistence performs so well in explaining the cross section of stock returns. Moreover, although they find tightness of financial constraints is related to a low persistence component of volatility and the business cycle is related to a high persistence component of volatility, they are not able to rationalize this association beyond their empirical results. The current paper provides theoretical foundations for all these issues, and in that sense complements their findings.
7 Conclusions

This paper intends to shed some light in the economics underlying a series of empirical patterns observed in the volatility of stock returns. With this aim, I introduced limited commitment into a general-equilibrium economy populated by investors with heterogeneous beliefs, and showed how the interaction of these frictions can help to explain some well known statistical properties of stock return volatility. In the setup of the model, I demonstrated that the state price density and, more importantly, the market prices of risk are composed by three risk factors: endowment risk, sentiment risk and solvency risk. The last factor, which scales up sentiment at times when solvency constraints bind is novel, and is key in understanding the empirical regularities in stock return volatility.

The model is consistent with volatility clustering or GARCH effects, because the three risk factors are persistent. Endowment risk is persistent because it is related to the investors endowment shares of aggregate endowment which fluctuate randomly between zero and one. Sentiment risk clusters because it relates with the investors optimal consumption share of aggregate endowment which also fluctuate randomly between zero and one. Solvency risk is persistent because it is driven by the shadow price of solvency constraints, which is related to the optimal exercise boundary of a sequence of American contingent claims in the excess utility of staying in the risk sharing agreement compared to choosing to default and switching to autarky forever. Because the shadow price of solvency constraints increases consumption just enough to enforce the solvency constraints, their binding exhibits persistence.

The model is also able to reproduce and identify the so-called transitory and permanent components of stock return volatility. Solvency risk is associated with the binding of solvency constraints which do not occur every instant. Therefore, as opposed to endowment and sentiment risk which are "Brownian motion type of risk", solvency risk is associated with lower frequency shocks or "Solvency constraint binding type of risk" which present a lower persistence, according to sustained periods of binding or non-binding of solvency constraints.

The model provides a potential explanation for the conflicting evidence regarding the correlation between stock expected return and stocks return volatility. Typical explanations are associated either with "leverage effects" or "volatility-feedback effects". I provide a novel explanation in this paper, which implies that the binding of solvency constraints generates simultaneously a decrease in stock return and an increase or decrease in the instantaneous market price of risk, depending on the direction of disagreement of the population facing limited commitment, which may be relatively optimistic (positive correlation) or relatively pessimistic (negative correlation).

Finally, the model provides a rationale for the empirical result that different volatility factors are significant in the cross section of stock returns. Sentiment and solvency risk are present in the equilibrium market prices of risk and correspond to different volatility factors. Since these are systematic risk factors, they also appear in the equilibrium consumption CAPM derived for several stocks. Hence, volatility factors are successful in explaining the cross section of stock returns because they contain information related to the fundamental risks affecting the economy.

Most of the results presented are shown to hold for the market price of risk which is a common component of the short-run stock return volatility. However, the characterization obtained for the long-run stock return volatility suggests that there may be interesting dynamics generated by solvency risk in the long-run stock return volatility also. Therefore, a deeper empirical analysis of this component is suggested as a topic for future research.
APPENDIX

Proof of optimal consumption for investor A. First note that at each time $t$, the solvency constraint is satisfied when:

$$V_t = \mathbb{E}_t^B \int_t^\infty \frac{\eta_u e^{-\rho(u-t)}}{\eta_t} \left( \frac{1}{\alpha} (c_{A,u}^\alpha - \delta_{A,u}^\alpha) \right) du \geq 0 \quad (A1)$$

which holds if and only if

$$\eta_t e^{-\rho t} V_t = \mathbb{E}_t^B \int_t^\infty \eta_u e^{-\rho u} \left( \frac{1}{\alpha} (c_{A,u}^\alpha - \delta_{A,u}^\alpha) \right) du \geq 0 \quad (A2)$$

Since (A1) and (A2) are equivalent, we use for simplicity the solvency constraint (A2).

The value function of this problem is:

$$J = \inf_{\beta_t} \sup_{c_{A,t}} \left\{ \mathbb{E}_0^B \int_0^\infty \left[ \eta_t e^{-\rho t} \left( \frac{1}{\alpha} (c_{A,t}^\alpha - \delta_{A,t}^\alpha) \right) \right] d\beta_t \right\}$$

where $\lambda_A$ is the constant Lagrangian multiplier of the static budget constraint (24) and $\beta_t$ is the time-varying Lagrangian multiplier of the dynamic solvency constraint (25). The following result holds,

**Lemma 1.** The last component of the value function can be expressed as:

$$\mathbb{E}_0^B \int_0^\infty \left[ \mathbb{E}_t^B \int_t^\infty \eta_u e^{-\rho u} \left( \frac{1}{\alpha} (c_{A,u}^\alpha - \delta_{A,u}^\alpha) \right) du \right] d\beta_t = \mathbb{E}_0^B \int_0^\infty \eta_t e^{-\rho t} \left( \frac{1}{\alpha} (c_{A,t}^\alpha - \delta_{A,t}^\alpha) \right) \beta_t dt$$

**Proof of Lemma 1.** Denote by $X_u$ the integrand in the left hand side:

$$X_u = \eta_u e^{-\rho u} \left( \frac{1}{\alpha} (c_{A,u}^\alpha - \delta_{A,u}^\alpha) \right)$$

We need to compute $\mathbb{E}_0^B \int_0^\infty (\mathbb{E}_t^B \int_t^\infty X_u du) d\beta_t$. We have that

$$\mathbb{E}_0^B \int_0^\infty \left( \mathbb{E}_t^B \int_t^\infty X_u du \right) d\beta_t = \mathbb{E}_0^B \int_0^\infty \left( \int_0^\infty X_u du \right) d\beta_t = \mathbb{E}_0^B \int_0^\infty \left( \int_0^u d\beta_t \right) X_u du = \mathbb{E}_0^B \int_0^\infty X_u \beta_u du$$

where the first equality follows from the law of iterated expectations, and the second equality follows from changing the order of integration. Recalling the definition of $X_u$, we get the result in the lemma.

Using Lemma 1 and grouping terms, we can write the value function as:

$$J(1 + \beta_t) = \inf_{\beta_t} \sup_{c_{A,t}} \mathbb{E}_0^B \int_0^\infty \left[ (1 + \beta_t) \eta_t e^{-\rho t} \left( \frac{1}{\alpha} (c_{A,t}^\alpha - \delta_{A,t}^\alpha) \right) - \lambda_A \xi_t^B (c_{A,t} - \delta_{A,t}) + \eta_t e^{-\rho t} \left( \frac{1}{\alpha} \delta_{A,t}^\alpha \right) \right] dt$$

Taking into account first the supremum we get from the first order conditions.

**Proof of Proposition 1.** Departing from equation (36), $\omega^\alpha (\eta_t, 1 + \beta_t)$ can be written as:
\[ \omega^\alpha(\eta_t, 1 + \beta_t) = \left\{ \left( 1 + \left[ \frac{\lambda_A}{\lambda_B} \left( \frac{1}{1 + \beta_t} \right) \right]^{1/\alpha} \right) + \left[ \frac{1}{\eta_t} \right]^{1/\alpha} - 1 \right\}^{-1} \left[ \frac{\lambda_A}{\lambda_B} \left( \frac{1}{1 + \beta_t} \right) \right]^{1/\alpha} \right\}^{-\alpha} \]  

(A3)

Next, from the definition of \( \theta(\beta_t) \) in the proposition we can write

\[ \theta(1 + \beta_t) = \left\{ 1 + \left[ \frac{\lambda_A}{\lambda_B} \left( \frac{1}{1 + \beta_t} \right) \right]^{1/\alpha} \right\}^{-1} \]

(A4)

Combining equations (A3) and (A4) we get:

\[ \omega^\alpha(\eta_t, 1 + \beta_t) = \theta^\alpha(1 + \beta_t) \left\{ 1 + [1 - \theta(1 + \beta_t)] \left[ \frac{1}{\eta_t} \right]^{1/\alpha} - 1 \right\}^{-\alpha} \]

Applying the binomial theorem we obtain:

\[ \omega^\alpha(\eta_t, 1 + \beta_t) = \theta^\alpha(1 + \beta_t) \sum_{j=0}^{\infty} \left( \binom{-\alpha}{j} \left[ 1 - \theta(1 + \beta_t) \right]^j \frac{1}{\eta_t} \right)^{1/\alpha} - 1 \]

Finally, applying the binomial theorem again, we obtain the result in the Proposition.

Proof of Proposition 2. The proof proceeds along the same steps as for the problem with positive liquid wealth constraints (see El Karoui and Jeanblanc-Pique (1998) and Detemple and Serrat (2003)). Denote a non-decreasing process \( \hat{q} \) and consider the class of candidate state price densities \( B_t(\hat{q}) \), which will be taken as exogenous due to competitive behavior. Departing from the proof of optimal consumption of investor \( A \) at the beginning of this appendix and plugging back optimal consumption into the value function leads to the following singular control problem:

\[ J(1 + \beta_t) = \inf_{\{\beta_t\}} \left\{ \mathbb{E}^B_0 \int_0^\infty f(1 + \beta_t) \, dt + \mathbb{E}^B_0 \int_0^\infty \left( \eta_t e^{-\rho t} \frac{1}{\alpha} \delta_{A,t} + \lambda_A \xi_t^B(\hat{q}) \delta_{A,t} \right) \, dt \right\} \]

where

\[ f(1 + \beta_t) = (1 + \beta_t) \eta_t e^{-\rho t} \left( \frac{1}{1 + \beta_t} \right)^{1/\alpha} - \delta_{A,t} \]

(A5)

The non-decreasing shadow price of solvency constraints \( \beta_t \) has a right continuous non-decreasing inverse \( \tau_u = \inf \{ t : \beta_t \geq u \} \). The processes \( \tau_u \) is a stopping time, and the following events coincide: \( \{ \beta_t \geq u \} = \{ \tau_u \leq t \} \). The following result holds:

Lemma 2. For any differentiable function \( f(q) \) we have that:

\[ \inf_{\{\beta_t\}} \mathbb{E}^B_0 \int_0^\infty f(1 + \beta_t) \, dt = \mathbb{E}^B_0 \int_0^\infty f(1) \, dt = \int_0^\infty \sup \mathbb{E}^B_0 \left( \int_0^\infty 1_{\{\tau_u \leq t\}} f'(1 + u) \, du \right) \]

Proof of Lemma 2. Let \( f(q) \) be a differentiable function. Using \( \{ \beta_t \geq u \} = \{ \tau_u \leq t \} \) we can write:
\[ f(1 + \beta_t) - f(1) = \int_0^{\beta_t} f'(1 + u) \, du \]
\[ = \int_0^\infty \mathbf{1}_{\{\beta_t \geq u\}} f'(1 + u) \, du \]
\[ = \int_0^\infty \mathbf{1}_{\{\tau_u \leq t\}} f'(1 + u) \, du \]

Integrating over time, taking the expectation and minimizing with respect to \( \beta_t \) (which is equivalent to maximize with respect to its inverse) yields the result.

Given the explicit function \( f(1 + \beta_t) \) in equation (A5) we can compute

\[ f(1) = \eta_t e^{-\rho t} \frac{1}{\alpha} \left[ \left( \frac{\lambda A}{\eta_t} e^{\rho t} \xi_t^B(\hat{q}) \right)^{-\frac{1}{1-\alpha}} - \delta_{A,t}^\alpha \right] - \lambda_A e^{\rho t} \xi_t^B(\hat{q}) \left( \frac{\lambda A}{\eta_t} e^{\rho t} \xi_t^B(\hat{q}) \right)^{-\frac{1}{1-\alpha}} \]
\[ f'(1 + \beta_t) = \eta_t e^{-\rho t} \frac{1}{\alpha} \left[ \left( \frac{1}{1 + \beta_t} \frac{\lambda A}{\eta_t} e^{\rho t} \xi_t^B(\hat{q}) \right)^{-\frac{1}{1-\alpha}} - \delta_{A,t}^\alpha \right] \]

Applying Lemma 2 we get:

\[ J(1 + \beta_t) - J(1) = \int_1^\infty \sup_{\hat{q}} E_0^B \left( \int_0^\infty \eta_t e^{-\rho t} \frac{1}{\alpha} \left[ \left( \frac{1}{1 + \beta_t} \frac{\lambda A}{\eta_t} e^{\rho t} \xi_t^B(\hat{q}) \right)^{-\frac{1}{1-\alpha}} - \delta_{A,t}^\alpha \right] dt \right) \, dq \]

(A6)

Let \( \hat{q}_q = \inf \{ t : 1 + \beta_t \geq q \} \) be the non-decreasing inverse of the non-decreasing process \( 1 + \beta_t \), and note that the following events are equivalent:

\[ \{ \tau_u \leq t \} = \{ \beta_t \geq u \} = \{ 1 + \beta_t \geq 1 + u \} = \{ \hat{q}_{1+u} \leq t \} \]

Next, we do the change of variables: \( q = 1 + u \) to obtain:

\[ \{ \tau_u \leq t \} = \{ \hat{q}_q \leq t \} \]

Performing this change of variables in equation (A6) we get:

\[ J(1 + \beta_t) - J(1) = 1_1^\infty \sup_{\hat{q}} E_0^B \left( \int_0^\infty \eta_t e^{-\rho t} \frac{1}{\alpha} \left[ \left( \frac{1}{q + \eta_t} \frac{\lambda A}{\eta_t} e^{\rho t} \xi_t^B(\hat{q}) \right)^{-\frac{1}{1-\alpha}} - \delta_{A,t}^\alpha \right] dt \right) \, dq \]

But in equilibrium it must hold that \( \xi^B(q) = \xi^B(\hat{q}) \), such that \( q = \hat{q} \) (right after the proof of this proposition, in Lemma 3 I show that such equilibrium exists and is unique). Considering this, and using the definitions given in the proposition we get:
exists a unique equilibrium where \( q = \hat{q} = q^* \), such that \( \xi_t^B(q) = \xi_t^B(\hat{q}) = \xi_t^B(q^*) \).

**Proof of Lemma 3.** First note that without assuming the existence of such equilibrium, all the results go through except that the expressions which currently depend on \( q \) only, will depend now on both \( q \) and \( \hat{q} \). In particular, we get that in this case

\[
\bar{\omega}(\eta_t, q, \hat{q}) = \frac{(\eta_t q)^{1/\alpha}}{(\eta_t \hat{q})^{1/\alpha} + \left( \frac{\lambda q}{\lambda \hat{q}} \right)^{1/\alpha}} = \left( \frac{q}{\hat{q}} \right)^{1/\alpha} \omega(\eta_t, \hat{q})
\]

and at the optimal boundary \( \eta_t = x_t \) it must now hold that

\[
F \left( \frac{q}{\hat{q}}, \eta_t, r_t \right) = \mathbb{E}_t^B \int_t^\infty \frac{\eta_u e^{-\rho(u-t)} 1}{\alpha} \delta^u_{A,u} \left[ \left\{ \left( \frac{q}{\hat{q}} \right)^{1/\alpha} \omega(\eta_u, \hat{q}) \right\}^\alpha (1 + r_u)^\alpha - 1 \right] 1_{\{\eta_u > x(q^* r_u)\}} du = 0
\]

In order to proof the lemma, we need to show that for the special case where \( q = \hat{q} = q^* \), there exists a unique \( q^* \) which solves \( F(1, q^*, \eta_t, r_t) = 0 \), with

\[
F(1, q^*, \eta_t, r_t) = \mathbb{E}_t^B \int_t^\infty \frac{\eta_u e^{-\rho(u-t)} 1}{\alpha} \delta^u_{A,u} \left[ \left\{ \left( \frac{q}{\hat{q}} \right)^{1/\alpha} \omega(\eta_u, \hat{q}) \right\}^\alpha (1 + r_u)^\alpha - 1 \right] 1_{\{\eta_u > x(q^* r_u)\}} du \tag{A7}
\]

By definition \( q^* \in [1, +\infty) \). It can be shown that as \( q^* \to +\infty \), \( \omega \to 1 \) and \( x(q^* r_u) \to 0 \). Therefore we have:

\[
\lim_{q^* \to \infty} F(1, q^*, \eta_t, r_t) = \mathbb{E}_t^B \int_t^\infty \frac{\eta_u e^{-\rho(u-t)} 1}{\alpha} \delta^u_{A,u} [(1 + r_u)^\alpha - 1] du < 0
\]

where the inequality follows from the fact that \( r > 0 \) and \( \alpha < 0 \)

Similarly when \( q^* \to 1 \) we get:

\[
\lim_{q^* \to 1} F(1, q^*, \eta_t, r_t) = \mathbb{E}_t^B \int_t^\infty \frac{\eta_u e^{-\rho(u-t)} 1}{\alpha} \delta^u_{A,u} \left[ \omega^\alpha(\eta_u, 1) (1 + r_u)^\alpha - 1 \right] 1_{\{\eta_u > x(1, r_u)\}} du > 0
\]

where the inequality follows from the definition of the optimal exercise boundary and the complementary slackness condition.

Finally, note from equation (A7) that when \( q^* \) increases both the argument in the expectation as well as the probability of the indicator been equal to one increase. The former increases because the expected utility of staying in the risk sharing agreement is increasing in \( q^* \). The later increases
because the optimal boundary is decreasing in $q^*$. Therefore, $F(1, q^*, \eta_t, r_t)$ is monotonically increasing in $q^*$.

It then follows that there exists a unique $q = \hat{q} = q^* \subset [1, +\infty)$ which satisfies $F(1, q^*, \eta_t, r_t) = 0$.

**Proof of Proposition 3.** We seek a new measure $Z$ such that under this measure

$$
\mathbb{E}_t^Z [X_u (\delta_{A,u})^\alpha] = \mathbb{E}_t^Z \left[ X_u \mathbb{E}_t^B [(\delta_{A,u})^\alpha] \right]
$$

We have that:

$$
\mathbb{E}_t^B [X_u (\delta_{A,u})^\alpha] = \mathbb{E}_t^Z \left[ \frac{dQ^B}{dQ} X_u (\delta_{A,u})^\alpha \right] = \mathbb{E}_t^Z \left[ X_u \mathbb{E}_t^B [(\delta_{A,u})^\alpha] \right]
$$

where the first equality follows by definition, and the second is imposed in order to eliminate $(\delta_{A,u})^\alpha$ from the expectation. It follows that

$$
\frac{dQ^Z}{dQ^B} = \left( \frac{\delta_{A,u}}{\delta_{A,u}} \right)^\alpha = e^{-\frac{1}{2} (\alpha \sigma_{\delta_{A,u}}^2) u + (\alpha \sigma_{\delta_{A,u}}) W_u^A}
$$

where the second equality comes from the fact that $(\delta_{A,u})^\alpha$ is a geometric Brownian Motion. Finally, the relationship between the Brownians under the two measures follows from Girsanov theorem.

**Proof of Proposition 4.** The equation for the optimal exercise boundary follows from Proposition 3. We seek the function $H(\ln \eta_t, \ln r_t, u - t, x(\theta, \cdot), \chi, \epsilon) = \mathbb{E}_t^Z [\eta^\chi_t(1 + r_u)^\epsilon \mathbf{1}_{\{\eta_u > x_u\}}]$. We note that

$$
\mathbb{E}_t^Z \left[ \eta^\chi_t(1 + r_u)^\epsilon \mathbf{1}_{\{\eta_u > x_u\}} \right] = \mathbb{E}_t^Z \left[ \mathbb{E}_t^Z \left[ \eta^\chi_t(1 + r_u)^\epsilon \mathbf{1}_{\{\eta_u > x_u\}} \right] \bigg| r_u \right]
$$

$$
= \mathbb{E}_t^Z \left[ \mathbb{E}_t^Z \left[ e^{\ln \eta^\chi_t} (1 + e^{\ln r_u})^\epsilon \mathbf{1}_{\{\ln \eta^\chi_t > \chi \ln x(\ln r_u)\}} \right] \bigg| r_u \right]
$$

From the dynamics under $Z$ of $\ln \eta^\chi_t$ and $\ln r_u$, it follows that they are jointly normal with means, variances and covariance given by $\mu_{\eta^\chi}$, $\mu_r$, $\sigma^2_{\eta^\chi}$, $\sigma^2_r$ and $\sigma_{\eta r}$ respectively, where:

$$
\mu_{\eta^\chi}(\ln \eta_t, \chi, u - t) = \chi \ln \eta_t - \left[ \alpha \chi \bar{\mu}_B \sigma_{\delta_A} + \frac{1}{2} \chi \left( \bar{\mu}_A^2 + \bar{\mu}_B^2 \right) \right] (u - t)
$$

$$
\mu_r(\ln r_t, u - t) = \ln r_t + \left[ \hat{\mu}_B^B - \hat{\mu}_A^B + (1 - \alpha) \sigma_{\delta_A}^2 - \frac{1}{2} (\sigma_{\delta_A}^2 + \sigma_{\delta_B}^2) \right] (u - t)
$$

$$
\sigma^2_{\eta^\chi}(\chi, u - t) = \chi^2 \left( \bar{\mu}_A^2 + \bar{\mu}_B^2 \right) (u - t)
$$

$$
\sigma^2_r(u - t) = \left( \sigma_{\delta_A}^2 + \sigma_{\delta_B}^2 \right) (u - t)
$$

$$
\sigma_{\eta r}(\chi, u - t) = \chi \left( \bar{\mu}_A \sigma_{\delta_A} - \bar{\mu}_B \sigma_{\delta_B} \right) (u - t)
$$

Therefore, the conditional distribution of $\ln \eta^\chi_t | r_u$ is normal with mean $\bar{\mu}$ and variance $\bar{\sigma}^2$, which are given by
\[
\pi \left( \ln \eta_t, \ln r_t, \ln r_u, \chi, u - t \right) = \mu_\eta \left( \ln \eta_t, \chi, u - t \right) + \frac{\sigma_{\eta r} \left( \chi, u - t \right)}{\sigma_f^2 \left( u - t \right)} \left( \ln r_u - \mu_r \left( \ln r_t, u - t \right) \right)
\]

\[
\sigma^2 \left( \chi, u - t \right) = \sigma_\eta^2 \left( \chi, u - t \right) \left[ 1 - \frac{\sigma_{\eta r}^2 \left( \chi, u - t \right)}{\sigma_r^2 \left( u - t \right) \sigma_\eta^2 \left( \chi, u - t \right)} \right]
\]

Using the properties of truncated lognormals, it follows that the function \( d \) is given by

\[
d \left( \ln \eta_t, \ln r_t, \ln r_u, x \left( \ln r_u \right), \chi, u - t \right) = \frac{\pi \left( \ln \eta_t, \ln r_t, \ln r_u, \chi, u - t \right) - \chi \ln x \left( \ln r_u \right)}{\sigma \left( \chi, u - t \right)}
\]

The expression for \( \beta_t \) follows from inverting \( \theta \left( \beta_t \right) \) using its definition in Proposition 1.

**Proof of Proposition 5.** The probability under the objective measure of the event \( \{ \eta_u < x \left( r_u \right) \} \) is given by

\[
P \left[ \eta_u < x \left( r_u \right) \right] = \int_{-\infty}^{+\infty} N \left[ \frac{\ln x \left( \ln r_u \right) - \mu_\eta^* \left( \ln \eta_t, u - t \right)}{\sqrt{\left( \nabla_\delta A \right)^2 + \left( \nabla_\delta B \right)^2}} \right] n \left( \frac{\ln r_u - \mu_r^* \left( \ln r_t, u - t \right)}{\sqrt{\left( \sigma_\delta A \right)^2 + \left( \sigma_\delta B \right)^2}} \right) d \left( \ln r_u \right)
\]

with

\[
\mu_\eta^* \left( \ln \eta_t, u - t \right) = \ln \eta_t - \left[ \beta_{\delta A} - \beta_{\delta A}^B \right] + \beta_{\delta B} - \beta_{\delta A} \left[ \left( \nabla_\delta A \right)^2 + \left( \nabla_\delta B \right)^2 \right] \left( u - t \right)
\]

\[
\mu_r^* \left( \ln r_t, u - t \right) = \ln r_t + \left[ \beta_{\delta B} - \beta_{\delta A} - \frac{1}{2} \left( \sigma_\delta B \right)^2 \right] \left( u - t \right)
\]

Taking the derivative with respect to \( \eta_t \) we get:

\[
\frac{\partial P \left[ \eta_u < x \left( r_u \right) \right]}{\partial \eta_t} = -\frac{1}{n} \int_{-\infty}^{+\infty} n \left[ \frac{\ln x \left( \ln r_u \right) - \mu_\eta^* \left( \ln \eta_t, u - t \right)}{\sqrt{\left( \nabla_\delta A \right)^2 + \left( \nabla_\delta B \right)^2}} \right] n \left( \frac{\ln r_u - \mu_r^* \left( \ln r_t, u - t \right)}{\sqrt{\left( \sigma_\delta A \right)^2 + \left( \sigma_\delta B \right)^2}} \right) d \left( \ln r_u \right) < 0
\]

which is strictly negative.

**Proof of Proposition 7.** As an example, here I derive the diffusion vector \( \sigma_{S.A} \) associated with the return on the financial security \( S_{A,t} \). I define the martingale \( M^A_t \)

\[
M^A_t = \int_0^\infty \mathbb{E}_0 \left[ \phi_{A,u}^B \right] du \quad (A8)
\]

and from the martingale representation theorem there exists an integrable adapted \((2x1)\) process \( \phi_A \) such that
\[ M_t^A = M_0^A + \int_0^\infty \phi_A dW_t^B \]  

(A9)

From equation (46) and equation (A8) we can express \( S_{A,t} \) as:

\[
S_{A,t} = \frac{1}{\xi_t} \left[ M_A^t - \int_0^t \xi_u^B \delta_{A,u} du \right]
\]

Applying Ito lemma considering the dynamics of the state price density from equation (20) as well as the dynamics of \( M_t^A \) from equation (A9), and identifying the diffusion coefficients we obtain

\[
\left[ \begin{array}{c}
\sigma S_A \\
\sigma u A 
\end{array} \right] = \xi_t^B + \int_t^\infty \xi_t^B \left[ \left( \xi_u^B \delta_{A,u} \right) \right] du
\]

(A10)

Next, from the Clark-Ocone formula from Malliavin calculus we identify \( \phi_A \) as

\[
\phi_A = \mathbb{E}_t^B \left[ \mathcal{D}_t \left( M_u^A \right) \right]
\]

\[
= \int_t^\infty \mathbb{E}_t^B \left[ \mathcal{D}_t \left( \xi_u^B \delta_{A,u} \right) \right] du
\]

\[
= \sigma S_A \int_t^\infty \mathbb{E}_t^B \left[ \xi_u^B \delta_{A,u} \right] du + \int_t^\infty \mathbb{E}_t^B \left[ \delta_{A,u} \mathcal{D}_t \left( \xi_u^B \right) \right] du
\]

(A11)

Finally, plugging (A11) into (A10) gives the required expression.

The Malliavin derivatives for the exogenous state variables follow immediately as they are all Geometric Brownian Motions. The Malliavin derivative for the endogenous state variable \( t \) deserves some special attention. Define \( v [\tau, u] \) as the last time in \( [\tau, u] \) at which the solvency constraint bound:

\[
v[\tau, u] = \sup \left\{ v \in [\tau, u] : \Phi (r_v) = \sup_{s \in [\tau, u]} \Phi (r_s) \right\}
\]

Then, from Exercise 1.2.11 and the proof of Proposition 2.1.3 in Nualart (1995), and from the proof of equations (15) and (29) in Detemple and Serrat (2003), it follows that:

\[
\frac{\mathcal{D}_t (\beta_u)}{(1 + \beta_u)} = (1 - \alpha) \left( \frac{1}{\Phi (r_v[t,u])} \left( 1 - \Phi (r_v[t,u]) \right) \right) 1_{(v[0,u] \geq t)} \mathcal{D}_t \left[ \Phi (r_v[t,u]) \right]
\]

with

\[
\mathcal{D}_t \left[ \Phi (r_u) \right] = \frac{\partial \Phi}{\partial r_v} \mathcal{D}_t (r_v) + \frac{\partial \Phi}{\partial \eta_v} \mathcal{D}_t (\eta_v)
\]

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with

\[
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\]

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with

\[
\mathcal{D}_t \left[ \Phi (r_u) \right] = \frac{\partial \Phi}{\partial r_v} \mathcal{D}_t (r_v) + \frac{\partial \Phi}{\partial \eta_v} \mathcal{D}_t (\eta_v)
\]
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38


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Table 1
Choice of parameter values and benchmark values for the state variables

This table lists the annualized parameter values used for all the figures in the paper. These values are consistent with those used by Dumas, Kurshev and Uppal (2008). The table also indicates the annualized benchmark values taken by the state variables except for the particular one varied in a given graph.

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Parameters for investors' endowments</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average growth rate of endowment of investor A</td>
<td>( \mu \delta_A )</td>
<td>0.08</td>
</tr>
<tr>
<td>Average growth rate of endowment of investor B</td>
<td>( \mu \delta_B )</td>
<td>0.05</td>
</tr>
<tr>
<td>Volatility of the growth rate of endowment of investor A</td>
<td>( \sigma \delta_A )</td>
<td>0.12</td>
</tr>
<tr>
<td>Volatility of the growth rate of endowment of investor B</td>
<td>( \sigma \delta_B )</td>
<td>0.10</td>
</tr>
<tr>
<td><strong>Parameters for investors' preferences and disagreement</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Disagreement about the average growth rate of endowment of investor A</td>
<td>( \frac{\bar{\mu} - \mu_A}{\bar{\mu} - \mu_B} )</td>
<td>0.25</td>
</tr>
<tr>
<td>Disagreement about the average growth rate of endowment of investor B</td>
<td>( \frac{\bar{\mu} - \mu_A}{\bar{\mu} - \mu_B} )</td>
<td>0.25</td>
</tr>
<tr>
<td>Subjective discount rate for both investors</td>
<td>( \rho )</td>
<td>0.10</td>
</tr>
<tr>
<td>Relative risk aversion for both investors</td>
<td>( 1 - \alpha )</td>
<td>2</td>
</tr>
<tr>
<td>Ratio of shadow prices for the static budget constraint</td>
<td>( \frac{\lambda_B}{\lambda_A} )</td>
<td>1</td>
</tr>
<tr>
<td><strong>Benchmark value for the state variables</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ratio of endowments:</td>
<td>( r )</td>
<td>1</td>
</tr>
<tr>
<td>Change from investor B to investor A subjective measure</td>
<td>( \eta )</td>
<td>1</td>
</tr>
<tr>
<td>Shadow price for the dynamic solvency contraint</td>
<td>( \beta )</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 2
EGARCH parameters and p-values obtained from MonteCarlo Simulation

This table contains the principal results of the estimation of a typical EGARCH(1,1) model:

\[
R_t = c + \sigma_t \varepsilon_t, \quad \varepsilon_t \sim N(0,1)
\]
\[
\ln \sigma_t^2 = c_0 + c_1 \frac{\varepsilon_{t-1}}{\sigma_t} + c_2 \frac{\varepsilon_{t-1}}{\sigma_t} + c_3 \ln \sigma_{t-1}^2
\]

The first column corresponds to an EGARCH(1,1) estimation using 11583 daily observations of returns on the S&P500 Composite Price Index, from January 1st, 1964 until May 23rd, 2008. The other three columns correspond to versions of the model presented in this paper, where parameters are defined in Table 1. The second column refers to the two trees model analyzed in this paper but without sentiment nor solvency risks. The third column is related to the case of two trees and sentiment risk, but without considering solvency risk. Finally, the last column corresponds to the full model with two trees, sentiment and solvency risks. The numbers in the second, third and fourth column are the median value for 10,000 simulations where in each simulation I constructed a sample of 11583 daily observations, consistent with the sample of the S&P500. The rows labeled "p-value" below the estimated EGARCH coefficients, report the results of a t-test for every coefficient being equal to zero and are based on consistent standard errors as calculated in Bollerslev and Wooldridge (1992). P-values indicating significance of the EGARCH coefficient at the 5% level are in bold.

<table>
<thead>
<tr>
<th></th>
<th>S&amp;P500 daily data (1963 - 2008)</th>
<th>Two-trees</th>
<th>Two trees and Sentiment</th>
<th>Two trees, sentiment and solvency</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>0.0003</td>
<td>0.0011</td>
<td>0.0013</td>
<td>0.0013</td>
</tr>
<tr>
<td>p-value</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>(c_0)</td>
<td>-0.1948</td>
<td>-5.8574</td>
<td>-0.0212</td>
<td>-0.0235</td>
</tr>
<tr>
<td>p-value</td>
<td>0.0000</td>
<td>0.0675</td>
<td>0.0068</td>
<td>0.0043</td>
</tr>
<tr>
<td>(c_1)</td>
<td>0.1160</td>
<td>0.0053</td>
<td>0.0071</td>
<td>0.0079</td>
</tr>
<tr>
<td>p-value</td>
<td>0.0000</td>
<td>0.2788</td>
<td>0.0150</td>
<td>0.0061</td>
</tr>
<tr>
<td>(c_2)</td>
<td>-0.0659</td>
<td>0.0008</td>
<td>-0.0029</td>
<td>-0.0039</td>
</tr>
<tr>
<td>p-value</td>
<td>0.0000</td>
<td>0.3357</td>
<td>0.0706</td>
<td>0.0265</td>
</tr>
<tr>
<td>(c_3)</td>
<td>0.9888</td>
<td>0.2512</td>
<td>0.9984</td>
<td>0.9981</td>
</tr>
<tr>
<td>p-value</td>
<td>0.0000</td>
<td>0.0485</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
This figure has two plots. The one in the left shows the optimal exercise boundary level $x(\theta, r_t)$ for (the log of) sentiment risk $\eta_t$, such that continuation in the risk sharing agreement is optimal if $\ln \eta_t > x(\theta, r_t)$. The optimal exercise boundary level $x(\theta, r_t)$ is plotted as a function of the share of aggregate consumption of investor $A$ that would obtain in the absence of sentiment risk, denoted $\theta$, and (the log of) the ratio of endowments $r_t = \frac{B_t}{A_t}$. The plot in the right shows the inverse of the optimal exercise boundary $x(\theta, r_t)$ with respect to $\theta$, which is denoted $\Phi(\cdot, r_t)$. It is shown as a function of (the log of) sentiment risk $\eta_t$ and (the log of) the ratio of endowments $r_t = \frac{B_t}{A_t}$.
Figure 2
Conditional short-run volatility

This figure plots short-run volatility $\sigma_{S_A}^{sr}$ against the ratio of endowments $r_t = \frac{\delta_{H,t}}{S_{A,t}}$, sentiment risk $\eta_t$ and solvency risk $\beta_t$. There are two curves in each plot: the dashed curve is for the case of full agreement across investors ($\bar{\pi}_{\delta_A} = \bar{\pi}_{\delta_H} = 0$) and the solid curve is for the case of disagreement across investors ($\bar{\pi}_{\delta_A} = \bar{\pi}_{\delta_H} = 0.25$). The parameter values used for the variables not considered in the plots are given in Table 1.
Figure 3
Unconditional short-run volatility and related variables

This figure shows the unconditional value of the state variables, and short-run stock returns and their volatility, all obtained by simulating the model for 46 years of daily data (11583 daily observations). The first row shows the realization of the ratio of endowments \( r_t \) and the optimal share of consumption of investor \( A, \omega (\eta_t, 1 + \beta_t) \). In the latter case, the solid line is for the case of full commitment \((\beta_t = 0 \text{ for all } t)\) and the dashed line is for the case of limited commitment \((\beta_t \text{ unrestricted})\). The second row plots the realization of solvency risk \( \theta ((1 + \beta_t) \) as well as the realization of the singular component \( dQ_t \). The plot on the left of the third row shows the realization of short-run volatility \( \sigma_{r_S}^2 A_t \). There are three curves in it: the solid curve is for the case of heterogeneous beliefs and limited commitment \((\beta_t \text{ and } \eta_t \text{ unrestricted})\); the dashed curve is for the case of heterogeneous beliefs and full commitment \((\beta_t = 0 \text{ for all } t \text{ and } \eta_t \text{ unrestricted})\) and the dotted curve is for the case of homogeneous beliefs and full commitment \((\beta_t = 0 \text{ and } \eta_t = 1 \text{ for all } t)\). The remaining three graphs plot, for the same simulation of shocks, respectively the difference between: (i) returns obtained assuming "two trees" minus returns obtained using only "one tree", (ii) returns obtained assuming "two trees and sentiment" minus returns using only "two trees", and (iii) returns obtained assuming "two trees, sentiment and solvency risk" minus returns obtained using only "two trees and sentiment".