

ON THE EQUIVALENCE OF FLOATING AND FIXED-STRIKE ASIAN OPTIONS

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Abstract

There are two types of Asian options in the financial markets which differ according to the role of the average price. We give a symmetry result between the floating and fixed-strike Asian options. The proof involves a change of numéraire and time reversal of Brownian motion. Symmetries are very useful in option valuation and in this case, the result allows the use of more established fixed-strike pricing methods to price floating-strike Asian options.

Keywords: Asian options, floating strike Asian options, put call symmetry, change of numéraire, time reversal, Brownian motion

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1. Introduction

The purpose of this paper is to establish a useful symmetry result between floating and fixed-strike Asian options. A change of probability measure or numéraire and a time reversal argument are used to prove the result for models where the underlying asset follows exponential Brownian motion.

There are many known symmetry results in financial option pricing. Such results are useful for transferring knowledge about one type of option to another and may be used to simplify coding of one type of option when the other is already coded. However, one must take care as the transformed option may not exist in the market or have a sensible economic interpretation.

These tricks become very useful for exotic options, when perhaps no closed form solution exists, but an equivalence relation holds, together with an accurate computational procedure for the related class. This is of particular interest for the Asian option since much is known about the fixed-strike case, but comparatively little work has been done for the floating-strike option.

The original result of this type dates back to Kruizenga [17] and Stoll [23]. This put-call parity relates European options with the same strikes and is true for general models of the stock price. Bates [3] derives relationships between European puts and calls with different strikes in models where the stock follows exponential Brownian motion. In fact, these results are more general, and Bartels [2] proves for time homogeneous diffusions,

$$p(x, K, r, \delta, t, T) = c(K, x, \delta, r, t, T)$$

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where p, c are the European put and call prices. Carr, Ellis and Gupta [5] prove a closely related result.

Such symmetry relationships hold for certain exotic options. Research by Carr and Chesney [4] derives relationships for American puts and calls with the same “moneyness” in a general diffusion setting. Equivalence between a Passport option and a fixed-strike Lookback option was shown in Delbaen and Yor [10] and Henderson and Hobson [14]. Lipton [20] obtained a relationship between Passport options and Asian options.

In contrast to the above, our results for Asian options relate two *different* types of arithmetic Asians, a floating-strike call (or put) and a fixed-strike put (or call) option. This is interesting, since from the parity shown by Alziary, Decamps and Koehl [1] involving all four Asians and a European put and call, it does not seem likely there would be such a result. It is simple, however to obtain a put-call parity between two Asians of the same type (say a floating call and put), also in Alziary et al. [1].

In common with the Black Scholes model and most literature on Asian options, we model the underlying asset (stock) by exponential Brownian motion. By choosing the stock price as numéraire, and a time reversal, the equivalence is derived. The tool of a numéraire change (or probability measure change) is very powerful when applied to option pricing, see Geman, El Karoui and Rochet [12].

The result holds for “forward starting” options, but not for the “in progress” case. It is well known that an “in progress” fixed-strike Asian may be written as a fixed-strike Asian when the averaging has not started, with a modified strike. This is not true however for a floating-strike option. However, this can be used to transform a fixed-strike “in progress” Asian to a floating strike option where the averaging has not yet begun.

Asian options have a payoff which depends on the average price of the underlying asset during some part of the life of the option. The average is usually arithmetic, and if the asset price is assumed to follow exponential Brownian motion, an explicit option price is not available as the arithmetic average of a set of lognormal distributions is not known explicitly. There are two types of Asian options - the fixed strike option, where the average relates to the underlying asset and the strike is fixed; and floating strike options where the average relates to the strike price.

Pricing of the fixed-strike Asian has been the subject of much research over the last ten years and academic interest in these options has experienced a revival recently, see Carr and Schröder [6], Donati-Martin, Ghomrasni and Yor [9]. Early work used the fact that the distribution of the geometric average of a set of lognormal distributions is also lognormal, see Conze and Visvanathan [8] and Turnbull and Wakeman [24]. A second popular line of research is to price the fixed-strike Asian by direct numerical methods, including Monte Carlo simulation (Kemna and Vorst [16], Lapeyre and Teman [18]) or numerically solving the PDE (Rogers and Shi [22], Alziary et al. [1]).

Geman and Yor [13] derive a Laplace transform of the price for the at-the-money and out-of-the-money fixed-strike call. Fu, Madan and Wang [11] compare numerical inversion of the Laplace transform with Monte Carlo and find the techniques complement each other. Numerical inversion can be unstable for low volatilities and short maturities, whereas Monte Carlo performs well in this case. These two methods used together provide very accurate approximate prices for the fixed-strike Asian option.

The floating-strike Asian option has received far less attention in the literature,

perhaps because the problem is more difficult in that the joint law of $\{S_t, A_t\}$ is needed. Levy [19], and Ritchken, Sankarasubramanian and Vijh [21] use various approximations based on joint lognormality of $\{S_t, A_t\}$, and Chung, Shackleton and Wojakowski [7] incorporate second order terms into the approximation of the couplet.

It is well known (Ingersoll [15], Rogers and Shi [22], Alziary et al. [1]) that the floating-strike Asian price satisfies a one dimensional pde, after a numéraire change. However this pde is awkward to solve numerically as the Dirac delta function appears as a coefficient. Vecer [25] recently used a change of numéraire and connections to passport options to derive a simpler one dimensional pde for both fixed and floating-strike options. Even so, pricing methods for floating strike options are underdeveloped compared with the more established methods for the fixed-strike option.

It is this fact which means a relationship between the prices of fixed and floating Asian options would be extremely useful. With such a connection, a floating-strike option could be priced using well known methods for the fixed-strike option.

The paper is structured as follows. Section 2 outlines the model and defines the fixed and floating Asian options. The next section gives the main result and proof.

2. The Model

We consider the standard Black Scholes economy with a risky asset (stock) and a money market account. We take as given a complete probability space (Ω, \mathcal{F}, P) with a filtration $(\mathcal{F})_{0 \leq t \leq \infty}$, which is right-continuous and such that \mathcal{F}_0 contains all the P -null sets of \mathcal{F} . We also assume the existence of a risk-neutral probability measure Q (equivalent to P) under which discounted asset prices are martingales, implying no arbitrage.

Under Q , the stock price follows

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t \quad (1)$$

where r is the constant continuously compounded interest rate, δ is a continuous dividend yield, σ is the instantaneous volatility of asset return and W is a Q -Brownian motion.

An Asian option contract is written at $t = 0$ and expires at $T > 0$. The arithmetic average A can be calculated at T given the price history from time $t_0 < T$. Let $D > 0$ denote the duration of averaging $D = T - t_0$. If $T = D$ the average is computed over the whole life of the option, termed “plain vanilla” option. If $T > D$ the option is “forward starting.” If $T < D$ the Asian option is “in progress.”

Defining the arithmetic average to be

$$A_t = \frac{1}{D} \int_{t_0}^t S_u du$$

where $t \geq 0$, then the final value of the average A_T can be calculated.

The fixed and floating Asian options are defined as follows. By arbitrage arguments, the time 0 price of a *fixed-strike call* c_x is given by:

$$c_x(K, S_0, r, \delta, 0, T) = c_x = e^{-rT} E(A_T - K)^+ \quad (2)$$

whilst the time 0 price of a *floating-strike call*, c_f is given by

$$c_f(S_0, \lambda, r, \delta, 0, T) = c_f = e^{-rT} E(\lambda S_T - A_T)^+. \quad (3)$$

with $\lambda = 1$ being the important case in financial option pricing. The floating-strike call is typically interpreted as a call written on S , with floating strike A_T . Exercising, the holder receives or buys λ units of stock and pays the average of past prices, A_T .

Asian put options are defined analogously via

$$p_x(K, S_0, r, \delta, 0, T) = p_x = e^{-rT} E(K - A_T)^+ \quad (4)$$

(fixed) and

$$p_f(S_0, \lambda, r, \delta, 0, T) = p_f = e^{-rT} E(A_T - \lambda S_T)^+. \quad (5)$$

(floating).

For the following, we assume $t_0 = 0$ and hence the option is vanilla. The results hold however for the forward starting case, for prices computed up to and including time t_0 .

3. A Symmetry between Floating-Strike and Fixed-Strike Asian options

We give the symmetry results in the following theorem. The first part gives a relationship between the prices in (3) and (4), the floating-strike Asian call and fixed-strike Asian put. The second result relates the fixed-strike call (2) and the floating-strike put (5). It follows from the first using put-call parity for Asian options.

Theorem 1. *Under the assumption that S follows exponential Brownian motion in (1), the following symmetry results hold:*

$$\begin{aligned} (A) \quad c_f(S_0, \lambda, r, \delta, 0, T) &= p_x(\lambda S_0, S_0, \delta, r, 0, T) \\ (B) \quad c_x(K, S_0, r, \delta, 0, T) &= p_f(S_0, \frac{K}{S_0}, \delta, r, 0, T) \end{aligned}$$

Remark 1. It is interesting to note that the roles of the interest rate r and dividend δ have been reversed in the symmetry results.

Proof. We prove (A) first. The floating-strike Asian call price expressed *in units of stock as numéraire* is

$$c_f^* \equiv \frac{c_f}{S_0} = \frac{e^{-rT}}{S_0} E \left[(\lambda S_T - A_T)^+ \right] = E \left[\frac{S_T e^{-rT}}{S_0} \frac{(\lambda S_T - A_T)^+}{S_T} \right]$$

By changing numéraire to S via

$$\frac{S_T e^{-rT}}{S_0 e^{-\delta T}} = e^{-\frac{\sigma^2}{2}T + \sigma W_T} = \frac{dQ^*}{dQ}$$

the measure Q^* is defined. Under Q^* , $W_t^* = W_t - \sigma t$ is a Brownian motion, using the Girsanov theorem. Moreover

$$\frac{(\lambda S_T - A_T)^+}{S_T} = (\lambda - A_T^*)^+$$

is the *terminal payoff in units of stock as numéraire*, where $A_T^* \equiv \frac{A_T}{S_T}$. Now we have

$$c_f^* = e^{-\delta T} E^* \left[(\lambda - A_T^*)^+ \right].$$

Remark 2. We see the roles of the underlying and exercise price have switched and the new exercise price is λ units of stock. This is a put option written on a new asset A^* .

Continuing we have

$$A_T^* = \frac{A_T}{S_T} = \frac{1}{T} \int_0^T \frac{S_u}{S_T} du = \frac{1}{T} \int_0^T S_u^*(T) du$$

where for $u \leq T$ we define an \mathcal{F}_T -measurable random variable

$$\begin{aligned} S_u^*(T) \equiv \frac{S_u}{S_T} &= \exp \left\{ - \left(r - \delta - \frac{1}{2} \sigma^2 \right) (T - u) - \sigma (W_T - W_u) \right\} \\ &= \exp \left\{ \left(r - \delta + \frac{1}{2} \sigma^2 \right) (u - T) + \sigma (W_u^* - W_T^*) \right\} \end{aligned}$$

using W_t^* , a Q^* Brownian motion.

Now note that if $\forall t \hat{W}_t \equiv -W_t^*$ is a reflected Q^* -Brownian motion starting at zero, then $W_u^* - W_T^* \stackrel{law}{=} \hat{W}_{T-u}$ and

$$A_T^* \stackrel{law}{=} \hat{A}_T \equiv \frac{1}{T} \int_0^T e^{\sigma \hat{W}_{T-u} + (r - \delta + \frac{1}{2} \sigma^2)(u - T)} du$$

Reversing time via variable change $s = T - u$ gives

$$\hat{A}_T = \frac{1}{T} \int_0^T e^{\sigma \hat{W}_s - (r - \delta + \frac{1}{2} \sigma^2)s} ds$$

Thus $S_u^*(T)$ are indeed log-normally distributed variates and $A_T^* \stackrel{law}{=} \hat{A}_T$ is a sum of such log-normally distributed variates. Thus

$$c_f^* = e^{-\delta T} E^* (\lambda - A_T^*)^+ = e^{-\delta T} E^* (\lambda - \hat{A}_T)^+$$

and the result (A) is proved.

Now to prove the second part, (B). Beginning this time with a fixed-strike call

$$c_x(K, S_0, r, \delta, 0, T) = e^{-rT} E(A_T - K)^+$$

(B) follows from put-call parity results. For the floating strike, it is known that

$$p_f(S_0, \lambda, r, \delta, 0, T) - c_f(S_0, \lambda, r, \delta, 0, T) = \frac{1}{(r - \delta)T} (e^{-\delta T} - e^{-rT}) S_0 - \lambda S_0.$$

The analogous result for fixed strike options is

$$c_x(K, S_0, r, \delta, 0, T) - p_x(K, S_0, r, \delta, 0, T) = \frac{1}{(r - \delta)T} (e^{-\delta T} - e^{-rT}) S_0 - e^{-rT} K.$$

Combining these and (A) gives the result (B). Of course, this could also be proved directly using a similar method to the first part.

Remark 3. The above results extend trivially to forward start options where the averaging period begins at $t_0 > 0$, and the option is priced at times up to t_0 . They do not extend to “in progress” Asians due to the extra term created by the average to date A_t , at time t . More specifically, the payoff for the “in progress” floating strike call can be written as

$$(\lambda S_T - A_T)^+ = (\lambda S_T - \frac{1}{T} \int_{t_0}^0 S_u du - \frac{1}{T} \int_0^T S_u du)^+$$

when pricing at time 0 and $t_0 < 0$. When we scale with S_T , the known term (at time 0) $\int_{t_0}^0 S_u du$ will no longer be a constant.

Remark 4. The floating-strike Asian call (under Q) is thus equivalent to a fixed-strike Asian put with strike normalized to λ (under Q^*), an option to sell the “new asset” A^* , to receive λ units of numéraire (stock S).

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