

Working paper 04-01
April 2004 – Revision July 2004

Conditions Ensuring the Separability of Asset Demand for All Risk-Averse Investors

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Abstract

We explore how the demand for a risky asset can be separated into an investment effect and a hedging effect by all risk-averse investors. This question has been shown to be complex when considered outside of the mean-variance framework. We restrict dependence among returns on the risky assets to quadrant dependence and find that the demand for one risky asset can be decomposed into an investment component based on the risk premium offered by the asset and a hedging component used against fluctuations in the return on the other risky asset. We also show that the class of quadrant dependent distributions is larger than that of two-fund separating distributions. This conclusion opens up the search for broader distributional hypotheses suitable to asset-pricing models. Examples are discussed.

Keywords: Portfolio choice, investment effect, hedging effect, quadrant dependence, two-fund separation, asset-pricing model, copulas.

JEL classification: D80, G10, G11, G12.

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The mean-variance model of portfolio choice has been used extensively to answer the following question: Under what conditions can the demand for one risky asset be decomposed into an investment part and a hedging part? (Markowitz (1952), Mossin (1973) and Huang and Litzenberger (1988)). Though commonly used, the mean-variance model imposes strong conditions either on preferences or on return distributions (i.e. quadratic utility function or elliptical distributions). The normal distribution has been challenged by many empirical studies (Fama (1965) and Zhou (1993)) and the quadratic utility function displays increasing absolute risk aversion (Arrow (1971)). More recently, Beaulieu, Dufour and Khalaf (2003) have shown that mean-variance efficiency is still rejected (though less frequently) when non-normal distributions are considered. They concluded that more research is needed to better identify the necessary and sufficient distribution hypotheses applicable to asset pricing models.

Merton (1971) characterizes optimal dynamic portfolio strategies and shows that time-varying investment opportunities result in an optimal portfolio with two facets: a mean-variance part and an intertemporal hedging part. Merton's result, which was obtained under the assumption of a Markovian diffusion process, has recently been extended to more general semi-martingales by Kramkov and Schachermayer (1999). These authors have not found however the qualitative result capable of dissociating the hedging component from the risk premium component for all risk-averse investors.

The main objective of this article is to provide conditions ensuring the separability of asset demand for all risk-averse investors. It also proposes a class of distribution functions which includes the two-fund separation distributions.

We introduce and describe a new form of risk dependence, namely quadrant dependence. This concept has been defined by Lehmann (1966). This form of non-linear dependence describes how two random variables behave together when they are simultaneously small (or large). One important property of quadrant dependence is that if (X_1, X_2) is positive (negative) quadrant dependent, then the covariance between X_1 and X_2 is positive (negative). However, the

converse is not true (Tong, 1980). Quadrant dependence generalizes regression dependence (Tukey, 1958).

Quadrant dependence has its interest in modeling dependent risks beyond the usual linear assumptions since it can take into account the simultaneous downside (upside) evolution of asset prices by introducing a natural hedging property. Quadrant dependence is of particular interest in risk management since it looks at the joint occurrence of large losses. In this article we shall show how quadrant dependence permits the decomposition of asset demand in a very natural manner. Our results open up the search for more general asset-pricing models, such as the copula representation, that can free the use of stochastic dependence from its connection with linear correlation.

The two-fund separation theorem is limited to a few classes of return distributions: multivariate normal distribution (Ross (1978)), elliptical distributions (Owen and Rabinovitch (1983) and Chamberlain (1983)), and linear conditional expectation distributions (Wei, Lee and Lee (1999)). Moreover, though these distributions imply separation, the converse may not be true. We shall show that the Ross mutual fund separation theorem implies the family of quadrant dependent distributions. We shall also provide an example of a joint distribution in the quadrant dependent family that is not a two-fund separating distribution. These results indicate that the class of quadrant dependent distributions is larger than that of two-fund separating distributions.

Section I presents our model of portfolio choice and introduces the concept of quadrant dependence (Lehmann (1966)). In this section, we also derive our main results related to the decomposition of portfolio weights into a mean part and a hedging part. In Section II, we establish formal links between quadrant dependence and mutual fund separation (see Elton and Gruber (2000) for a recent article on two-fund separation). We also provide examples of quadrant dependent distributions for applications in finance. Section III concludes the article.

I. Characterizing optimal portfolios

I.1 Basic Model

We consider a risk-averse agent who allocates his wealth (normalized to one) between one risk-free asset (with return x_0) and two risky assets with returns \tilde{x}_i , for $i = 1, 2$. We denote the joint distribution function as $dF(x_1, x_2)$. We also note $[\underline{x}_1, \bar{x}_1]$ and $[\underline{x}_2, \bar{x}_2]$ as the supports for X_1 and X_2 , respectively, and α_i , $i = 0, 1, 2$, as the investment in asset i chosen so as to maximize expected utility in a world with unlimited short-selling and under the constraint that $\alpha_0 + \alpha_1 + \alpha_2 = 1$. The agent's random end-of-period wealth W is then equal to

$$W(\alpha_1, \alpha_2) = 1 + x_0 + \alpha_1(X_1 - x_0) + \alpha_2(X_2 - x_0).$$

We define E as the expectation operator and m_i as the risk premium associated with asset i , that is $m_i = E(X_i) - x_0$, for $i = 1, 2$. As usual, $u(\cdot)$ is the individual's von-Neumann-Morgenstern utility function which we assume to be increasing, concave in final wealth, and continuously differentiable to the second order. This last assumption is for convenience and is not necessary to derive our results. So the optimal portfolio is obtained by maximizing $Eu(W(\alpha_1, \alpha_2))$ with respect to α_1 and α_2 .

In the case of independence among the risky assets, the first-order condition of the maximization program with respect to α_1 , evaluated at $\alpha_1 = 0$, can be written as

$$(E(X_1) - x_0) E(u'(1 + x_0 + \alpha_2(X_2 - x_0))),$$

which has the sign of the risk premium associated with X_1 . It follows that α_1^* is positive, if and only if m_1 is positive, that is if and only if X_1 offers a positive risk premium. The same logic applies to α_2^* .

Allowing for dependence among returns on risky assets will make it more difficult to characterize the optimal portfolio. As an illustration, we consider, for a moment, the case of

mean-variance preferences; to be precise, we suppose $u(W) = W - \frac{b}{2}W^2$, where b is a positive parameter that captures the agent's risk aversion. We also assume the following regularity condition on the first derivative $u'(W) = 1 - bW > 0$ for all W . The explicit solution to the maximization problem yields:

$$\alpha_1^* = \frac{1 - b(1 + x_0)}{b} \frac{m_1 \sigma_{22} - m_2 \sigma_{12}}{\Delta},$$

$$\alpha_2^* = \frac{1 - b(1 + x_0)}{b} \frac{m_2 \sigma_{11} - m_1 \sigma_{12}}{\Delta},$$

where $\sigma_{ij} = Cov(X_i, X_j)$, $1 - b(1 + x_0) > 0$ from the regularity condition and $\Delta = m_2^2 \sigma_{11} + m_1^2 \sigma_{22} - 2m_1 m_2 \sigma_{12} + \sigma_{11} \sigma_{22} - \sigma_{12}^2 > 0$ from the second-order condition. It is easily observed that α_2^* , the optimal investment in asset 2, is a function of m_2 and of σ_{12} .

α_2^* can be decomposed into

$$\alpha_{2h}^* = -\frac{\sigma_{12}}{\sigma_{22}} \alpha_1^* = -\frac{1 - b(1 + x_0)}{b} \frac{\sigma_{12}}{\sigma_{22}} \frac{m_1 \sigma_{22} - m_2 \sigma_{12}}{\Delta},$$

the hedging part, and

$$\alpha_{2m}^* = \frac{1 - b(1 + x_0)}{b} \frac{1}{\sigma_{22}} \frac{\sigma_{11} \sigma_{22} - (\sigma_{12})^2}{\Delta} m_2 = km_2,$$

the investment part. Since k is strictly positive, α_{2m}^* is proportional to m_2 and $Sign(\alpha_1^* \alpha_{2h}^*) = -Sign(\sigma_{12})$. In the next section, we show how the set of return distributions proposed in this article can be used to obtain such separability for all risk-averse investors.

Since quadrant dependence is symmetric a similar decomposition can be obtained for α_1^* .

I.2 Quadrant Dependence

Definition 1 (Lehman, 1966): Let (X_1, X_2) be a bivariate random variable. We say that (X_1, X_2) is positively quadrant dependent (PQD, in short) if

$$P(X_1 \leq x_1, X_2 \leq x_2) \geq P(X_1 \leq x_1)P(X_2 \leq x_2) \text{ for all } x_1, x_2. \quad (1)$$

The dependence is strict if inequality holds for at least some pair (x_1, x_2) . Similarly, (X_1, X_2) is negatively quadrant dependent if (1) holds with the inequality sign reversed.

Intuitively, X_1 and X_2 are PQD if the probability that they are simultaneously small (or simultaneously large) is at least as great as it would be were they independent. PQD is invariant under strictly increasing transformations of the random variables.

Definition (1) can be equivalently written as

$$P(X_1 \leq x_1 / X_2 \leq x_2) \geq P(X_1 \leq x_1) \text{ for all } x_1, x_2.$$

Under this form, PQD expresses the fact that knowledge of X_2 being small increases the probability of X_1 being small.

PQD is in particular satisfied when random variables are associated (see Milgrom and Weber, 1982, for definition and application to auction theory). PQD is also fulfilled if (X_1, X_2) shows positive likelihood ratio dependence (PLRD, in short. See Lehmann, 1966). PLRD is obtained by requiring that the conditional density of X_1 , given X_2 , is monotonic. The bivariate normal density is an example of PLRD.

Example 1. A quadrant dependent distribution. Let $X_1 = a + bX_2 + U$, where X_2 and U are independent. Then (X_1, X_2) is positively or negatively quadrant dependent as $b \geq$ or ≤ 0 . In particular, the components of a bivariate normal distribution will be positively or negatively quadrant dependent according to whether the correlation coefficient is positive or negative (Lehmann (1966)).

Other examples will be presented in the next section. Under the assumption of quadrant dependence, we are able to establish our main result:

Proposition 1: *Let (X_1, X_2) be quadrant dependent, and let (α_1^*, α_2^*) be the optimal portfolio, then α_i^* can be decomposed for all risk averse investors as $\alpha_i^* = \alpha_{im}^* + \alpha_{ih}^*$, for $i = 1, 2$ with*

- a) $\alpha_{im}^* \geq 0$ if and only if $E(X_i) \geq x_0$, and
- b) $Sign(\alpha_j^* \alpha_{ih}^*) = -Sign(Cov(X_1, X_2))$, for $j \neq i$.

Proof: See the Appendix.

α_{im}^* and α_{ih}^* , for $i=1,2$, designate respectively, the investment part and the hedging part of asset i demand. The investment term depends on the risk premium offered by the risky asset, and the hedging term is a function of fluctuations in the return on the other risky asset. The intuition behind Proposition 1 is natural and a significant implication of the proposition is that we need only know the sign of the covariance to sign the hedging effect, even if we do not restrict our analysis to the mean-variance model.

One corollary from Proposition 1 is that the optimal positions (long vs. short) on the investment component (α_{im}^*) and the hedging component (α_{ih}^*) will depend solely on the distributions of the risky assets for all risk-averse investors. Preferences determine the trade-off between the risk premium effect and the hedging effect and set the total investment of the risky asset. The result of Proposition 1 is related to that of Ross (1978) who presented separation conditions that allow the

optimal portfolio to exhibit two-fund separation for all risk-averse investors. In Example 2 below, we show how two-fund separating distributions are related to quadrant dependent distributions.

In the next proposition we look at the situations where one asset has a zero risk premium or where the assets returns are not correlated. We have the next result.

Proposition 2: *Let (X_1, X_2) be quadrant dependent, then for all risk-averse investors and $i=1,2$:*

$\alpha_{ih}^* = 0$ if and only if $Cov(X_1, X_2) = 0$, and

$\alpha_{im}^* = 0$ if and only if $E(X_i) - x_0 = 0$.

Proof: See the Appendix.

In the particular case where one risky asset has a zero risk premium (we can interpret this asset as a derivative), Proposition 2 shows that a risk-averse investor may invest money in a risky asset even though there is no risk premium attached. The reason is that financial risks, as opposed to insurable risks, cannot be eliminated through pooling. It is however possible to reduce financial risk by investing in a correlated risky asset. The returns on this security may display either a strong positive or negative correlation with the basic asset. In either case, it is possible to reduce risk by taking an appropriate position in the derivative instrument.

To complete the characterization of the optimal financial portfolio, we now proceed to identify the different positions (long vs. short) that the investor will take on one risky asset if the other risky asset has a zero risk premium. As we already know, when an agent is allocating his wealth between a risk-free asset and one risky asset or when the two risky assets have independent returns, a positive risk premium is necessary and sufficient to obtain a positive investment. In the next proposition we generalize this result.

Proposition 3: Let (X_1, X_2) be quadrant dependent. If $m_j = 0, j = 1, 2$, then for all risk-averse investors, $\alpha_i^* \geq 0$ if and only if $E(X_i) \geq x_0, i = 1, 2$ and $i \neq j$. In this case, the position to take on X_j (long vs. short) will depend on the covariance between X_1 and X_2 .

Proof: See the Appendix.

Note that since a nil covariance is equivalent to independence in the class of quadrant dependent distributions (Lehmann (1966)), the position on $X_i, i = 1, 2$ will also depend on its risk premium, if the covariance between X_1 and X_2 is nil.

II. Examples

We now discuss additional examples of quadrant dependent distributions (for other examples see Lehmann (1959) and Tong (1980)).

Example 2. The second example is related to the set of distributions that allow for two-fund separation as defined by Ross (1978). We now show that this set is included in the broader set of quadrant dependent distributions. We know from Ross (1978) that, under two-fund separation, X_1 and X_2 can be written as:

$$X_i = x_0 + \beta_i(X_m - x_0) + U_i, \text{ for } i = 1, 2$$

where X_m is the return on the risky fund (or on any index) and U_i is a random variable such that $E(U_i) = Cov(U_i, X_m) = 0$. Moreover, for two-fund separation to hold, we must verify the necessary and sufficient condition that $E(U_i / X_m) = 0, i = 1, 2$.

The conditional distribution function of X_1 given $X_2 = x_2$ is given by

$$F(x_1 / X_2 = x_2) = \Pr(U_1 - \frac{\beta_1}{\beta_2} U_2 \leq x_1 - x_0 - \frac{\beta_1}{\beta_2} (x_2 - x_0)).$$

As we can see, $F(x_1 / X_2 = x_2)$ is always monotone in x_2 and the sign of this monotonicity depends on those of β_1 and β_2 which represent the sensitivity of each asset with respect to X_m .

If we apply the result of Proposition 1, we find $Sign(\alpha_1^* \alpha_{2h}^*) = Sign(\beta_1 \beta_2)$.

Example 2 shows that two-fund separating distributions generate quadrant dependence. An interesting question is the following: Can quadrant dependence only be satisfied by a two-fund separating distribution? The next example addresses this question and shows that quadrant dependence does not imply two-fund separation.

Example 3. We consider a simple case with three states of the world: The corresponding returns are -3, 1 and 3 for the second risky asset and -2, 1 and 2 for the first risky asset. We assume a zero risk-free interest rate. Table I gives the joint density of the returns. Quadrant dependence can be proven (Lehmann, 1966). It is easily seen that the rate of return on one risky asset cannot be expressed as a linear combination of the rate of return on the other risky asset.

(Table I about here)

We consider two risk-averse investors with preferences given respectively by $u_1(W) = W - \frac{1}{4}W^2$

(with $1 - \frac{1}{2}W > 0$ for all W), and

$$u_2(W) = \begin{cases} W - 1 & \text{if } W \leq 1 \\ \frac{1}{3}(W - 1) & \text{if } 1 \leq W \leq 2. \\ \frac{1}{3} & \text{if } W \geq 2. \end{cases}$$

Note that investor u_2 is not in the Cass-and-Stiglitz (1970) family of separating functions. Otherwise, we would always have separation. The optimal investments in the two risky assets for investor u_1 are $(8/3, -5/3)$. The optimal choice for investor u_2 is given by the semi-line $2\alpha_1^* + 3\alpha_2^* = 0$ and $\alpha_1^* + \alpha_2^* \geq 1$; this does not include the optimal choice for investor u_1 , as illustrated in Figure 1 (see Appendix for details). Two-fund separation is then not allowed by the distribution provided in Table I.

(Figure 1 about here)

Example 4. Suppose that (X_1, X_2, \dots, X_s) have a multinomial distribution corresponding to n trials and success probabilities (p_1, p_2, \dots, p_s) . For $i, j=1, 2, \dots, s$, (X_i, X_j) is PQD (Lehmann, 1966).

Other examples. The Cauchy distribution (given that $x_2 \in [0, 1]$) is a positive quadrant dependent distribution. The main difference between the normal distribution and the Cauchy distribution is the longer and flatter tails of the latter. Other examples of a negative quadrant dependent distribution are the bivariate Dirichlet and the bivariate hypergeometric. The Dirichlet extends the beta distribution to multivariate distributions. Finally, as shown by Tong (1980), quadrant dependence can be used to define dependence involving a mixture of distributions. The multivariate t is an example.

III. Conclusion

We have proposed the concept of quadrant dependence (Lehmann, 1966) to analyze portfolio choice. This concept describes how two random variables behave together when they are simultaneously small or large. By assuming that the returns on risky assets are quadrant dependent, we were able to decompose the demand for one risky asset into an investment part based on the risk premium offered by the asset and a hedging part used against fluctuations in the return on the other risky asset. This characterization of the optimal portfolio was done for all risk-averse investors. Quadrant dependence was shown to be less restrictive than two-fund separating distributions (Ross (1978)). These results open up the search for broader asset-pricing models, such as the copula representation, that can free stochastic dependence from its connection with linear correlation.

Several extensions of our article are possible. For example, we may look at orthant dependent distributions. Orthant dependence generalizes the bivariate notion of quadrant dependence to higher dimensions. Intuitively, as for PQD, X_1, X_2, \dots, X_n are positive orthant dependent if they are more likely to have large values as they would be were they independent. A natural and significant extension to our framework would be to verify whether orthant dependence can result in a similar decomposition between the investment component and the hedging component for portfolios with more than two risky assets.

Denuit and Scaillet (2001) provided two-test procedures for positive quadrant dependence. These procedures are closely related to those proposed by Davidson and Duclos (2000). These procedures did not reject the positive quadrant dependence among data for US and Danish insurance claims. Mimouni (2002) applied the two-test procedures to data on financial assets and found that positive quadrant dependence was not rejected. Further developments of these tests, for portfolios containing many stocks and derivatives, are open for future research.

Appendix

Proof of Proposition 1: Since the problem is symmetric we only prove the decomposition for α_2^* . Also, for the presentation, we suppose $\alpha_1^* \geq 0$ and we restrict our analysis to PQD. The proof for negative quadrant dependence and $\alpha_1^* \leq 0$ is similar.

The first-order condition with respect to α_2 can be written as

$$\begin{aligned} \frac{\partial}{\partial \alpha_2} E(u(W(\alpha_1, \alpha_2))) &= \int_{\bar{x}_2}^{\bar{x}_2} \int_{\bar{x}_1}^{\bar{x}_1} (x_2 - E(X_2)) u'(W(\alpha_1, \alpha_2)) dF(x_1, x_2) \\ &\quad + m_2 \int_{\underline{x}_2}^{\underline{x}_2} \int_{\underline{x}_1}^{\underline{x}_1} u'(W(\alpha_1, \alpha_2)) dF(x_1, x_2). \end{aligned}$$

Let α_{2h}^* be the solution to

$$E((X_2 - E(X_2)) u'(W(\alpha_1^*, \alpha_{2h}^*))) = Cov(X_2, u'(W(\alpha_1^*, \alpha_{2h}^*))) = 0. \quad (A1)$$

The remainder of the proof is done in two folds. First we prove that (A1) cannot have a positive solution, if it has any; and second, we prove the existence of a solution to (A1).

By (A1), the first order condition can be rewritten as

$$\frac{\partial}{\partial \alpha_2} E(u(W(\alpha_1^*, \alpha_{2h}^*))) = m_2 \int_{\underline{x}_2}^{\underline{x}_2} \int_{\underline{x}_1}^{\underline{x}_1} u'(W(\alpha_1^*, \alpha_{2h}^*)) dF(x_1, x_2).$$

It follows from the concavity of the objective function that $\alpha_2^* \geq \alpha_{2h}^*$ if and only if $m_2 \geq 0$, or that $\alpha_2^* - \alpha_{2h}^*$ has the same sign as m_2 . Defining $\alpha_{2m}^* = \alpha_2^* - \alpha_{2h}^*$ ends the proof of part a).

We now prove part *b*).

We use the following property that follows from positive quadrant dependence:

$$\text{Cov}(f(X_1), g(X_2)) \geq 0 \text{ for all nondecreasing functions } f \text{ and } g. \quad (\text{P})$$

For $\alpha_{2h} > 0$, (A1) can be written as

$$\begin{aligned} \int_{\underline{x}_2}^{\bar{x}_2} \int_{\underline{x}_1}^{\bar{x}_1} (x_2 - E(X_2)) u'(W(\alpha_1, \alpha_{2h})) dF(x_1, x_2) &= \int_{\underline{x}_2}^{E(X_2)} \int_{\underline{x}_1}^{\bar{x}_1} (x_2 - E(X_2)) u'(W(\alpha_1^*, \alpha_{2h})) dF(x_1, x_2) \\ &\quad + \int_{E(X_2)}^{\bar{x}_2} \int_{\underline{x}_1}^{\bar{x}_1} (x_2 - E(X_2)) u'(W(\alpha_1^*, \alpha_{2h})) dF(x_1, x_2) \\ &< \int_{\underline{x}_2}^{E(X_2)} \int_{\underline{x}_1}^{\bar{x}_1} (x_2 - E(X_2)) u'(1 + x_0 + \alpha_1^*(x_1 - x_0) + \alpha_{2h}(E(X_2) - x_0)) dF(x_1, x_2) \\ &\quad + \int_{E(X_2)}^{\bar{x}_2} \int_{\underline{x}_1}^{\bar{x}_1} (x_2 - E(X_2)) u'(1 + x_0 + \alpha_1^*(x_1 - x_0) + \alpha_{2h}(E(X_2) - x_0)) dF(x_1, x_2) \\ &= \int_{\underline{x}_2}^{\bar{x}_2} \int_{\underline{x}_1}^{\bar{x}_1} (x_2 - E(X_2)) u'(1 + x_0 + \alpha_1^*(x_1 - x_0) + \alpha_{2h}(E(X_2) - x_0)) dF(x_1, x_2) \leq 0. \end{aligned} \quad (\text{A2})$$

The first inequality follows from the concavity of u and $\alpha_{2h} \geq 0$; the second inequality follows from property (P) with $f(x_1) = -u'(1 + x_0 + \alpha_1^*(x_1 - x_0) + \alpha_{2h}(E(X_2) - x_0))$ and $g(x_2) = x_2 - E(X_2)$. The solution to (A1) is then certainly negative since for $\alpha_{2h} > 0$ $E((\tilde{x}_2 - E(\tilde{x}_2)) u'(W(\alpha_1^*, \alpha_{2h})))$ is strictly negative.

To prove the existence of α_{2h}^* we make use of the theorem of the intermediate value. By the continuity of $E((X_2 - E(X_2)) u'(W(\alpha_1^*, \alpha_{2h})))$ in α_{2h} we will be done if we prove that there exists a $\bar{\alpha}_{2h}$ where $E((X_2 - E(X_2)) u'(W(\alpha_1^*, \bar{\alpha}_{2h})))$ is positive (we already know from (A2) that $E((X_2 - E(X_2)) u'(W(\alpha_1^*, \alpha_{2h})))$ takes negative values).

After integration by parts, (A1) simplifies to:

$$-\int_{\underline{x}_2}^{\bar{x}_2} \theta(x_2) K'(x_2) dx_2,$$

where $\theta(x_2) = \int_{\underline{x}_2}^{x_2} (t - E(X_2)) dG(t)$, $K(x_2) = \int_{\underline{x}_1}^{\bar{x}_1} u'(1 + \alpha_1^*(x_1 - x_0) + \alpha_{2h}(x_2 - x_0)) dF(x_1 / x_2)$ and $dF(x_1, x_2) = dF(x_1 / x_2) dG(x_2)$.

Since $\theta(\cdot)$ is negative, to complete the proof we need to show that $K(\cdot)$ is positive for a given value of α_{2h} .

Let n a positive integer and replace α_{2h} by $-n\alpha_1^*$ in $K(x_2)$. $K'(x_2)$ simplifies to

$$K'(x_2) = -\alpha_1^* \int_{\underline{x}_1}^{\bar{x}_1} u''(1 + \alpha_1^*(x_1 - x_0) - n\alpha_1^*(x_2 - x_0)) \left[n \frac{\partial}{\partial x_1} F(x_1 / x_2) + \frac{\partial}{\partial x_2} F(x_1 / x_2) \right] dx_1.$$

By continuity, and since $[\underline{x}_1, \bar{x}_1]$ and $[\underline{x}_2, \bar{x}_2]$ are compact and $\frac{\partial}{\partial x_1} F(x_1 / x_2)$ is positive, there

exists at least one \bar{n} where $\bar{n} \frac{\partial}{\partial x_1} F(x_1 / x_2) + \frac{\partial}{\partial x_2} F(x_1 / x_2) \geq 0$ for all $x_1, x_2 \dots$. As a result, for $\alpha_{2h} = -\bar{n}\alpha_1^*$, $K'(x_2)$ is positive for all $x_2 \in [\underline{x}_2, \bar{x}_2]$ and hence

$$E \left((X_2 - E(X_2)) u' \left(1 + \alpha_1^*(x_1 - x_0) - \alpha_1^* \bar{n} (x_2 - x_0) \right) \right) \geq 0.$$

This ends the proof of the existence of α_{2h}^* .

Proof of Proposition 2: By symmetry we only prove the result for α_2^* . We write the joint distribution of (X_1, X_2) as $dF(x_1/x_2)dG(x_2)$. We know that if $Cov(X_1, X_2) = 0$ then $\alpha_{2h}^* = 0$. It remains to show that if $\alpha_{2h}^* = 0$ then the two random variables have a nil covariance. Integrating by parts the left-hand-side term in (A1) yields

$$-\int_{x_2}^{\bar{x}_2} \left(\int_{x_2}^{x_2} (t - E(X_2)) dG(t) \right) \left(\int_{x_1}^{\bar{x}_1} u'(W(\alpha_1^*, 0)) \frac{\partial}{\partial x_2} dF(x_1/x_2) \right) dx_2 = 0. \quad (A3)$$

Under our assumption of quadrant dependence and since $\int_{x_2}^{x_2} (t - E(X_2)) dG(t) \leq 0$ for all x_2 , in order for equality in (A3) to hold, we need to have

$$\frac{\partial}{\partial x_2} F(x_1/x_2) = 0 \text{ for all } x_2,$$

which means that X_1 and X_2 have a nil covariance.

Part b) of the proposition follows from Proposition 1. *Q.E.D.*

Proof of Proposition 3: Proving the first part of Proposition 3 is equivalent to proving that

$$Sign(\alpha_1^*) = Sign(m_1).$$

Since the agent is risk averse he will always prefer the certainty equivalent to a gamble with the same expected return. In fact, with Jensen's inequality, one has

$$\begin{aligned} E(u(W(\alpha_1^*, 0))) &\leq u(E(W(\alpha_1^*, 0))) \\ &= u(1 + x_0 + m_1 \alpha_1^*). \end{aligned}$$

If α_1^* and m_1 have opposite signs then $m_1\alpha_1^* < 0$ and hence

$$E(u(W(\alpha_1^*, 0))) < u(1+x_0).$$

The latter inequality contradicts the optimality of $(\alpha_1^*, 0)$ since $(0, 0)$ is a better investment strategy. Consequently, $m_1 \geq 0$ is necessary and sufficient to obtain $\alpha_1^* \geq 0$. In addition, from Proposition 2, and since $m_2 = 0$, we know that $\alpha_{2m}^* = 0$. The optimal position to take on X_2 is then given by part b) of Proposition 1. *Q.E.D.*

Example 3: Since x_0 is normalized to 0, the random end-of-period wealth is $W(\alpha_1, \alpha_2) = 1 + \alpha_1 X_1 + \alpha_2 X_2$.

From Table 1, the expected utility function of the second agent for an investment (α_1, α_2) is

$$E(u_2) = \frac{1}{6}u_2(1-2\alpha_1-3\alpha_2) + \frac{1}{2}u_2(1+\alpha_1+\alpha_2) + \frac{1}{3}u_2(1+2\alpha_1+3\alpha_2),$$

where

$$u_2(1-2\alpha_1-3\alpha_2) = \begin{cases} -2\alpha_1-3\alpha_2 & \text{if } 2\alpha_1+3\alpha_2 \geq 0 \\ \frac{1}{3}(-2\alpha_1-3\alpha_2) & \text{if } -1 \leq 2\alpha_1+3\alpha_2 \leq 0 \\ \frac{1}{3} & \text{if } 2\alpha_1+3\alpha_2 \leq -1. \end{cases}$$

$$u_2(1+\alpha_1+\alpha_2) = \begin{cases} \alpha_1+\alpha_2 & \text{if } \alpha_1+\alpha_2 \leq 0 \\ \frac{1}{3}(\alpha_1+\alpha_2) & \text{if } 0 \leq \alpha_1+\alpha_2 \leq 1 \\ \frac{1}{3} & \text{if } \alpha_1+\alpha_2 \geq 1. \end{cases}$$

$$u_2(1+2\alpha_1+3\alpha_2) = \begin{cases} 2\alpha_1+3\alpha_2 & \text{if } 2\alpha_1+3\alpha_2 \leq 0 \\ \frac{1}{3}(2\alpha_1+3\alpha_2) & \text{if } 0 \leq 2\alpha_1+3\alpha_2 \leq 1 \\ \frac{1}{3} & \text{if } 2\alpha_1+3\alpha_2 \geq 1. \end{cases}$$

We have 12 different scenarios for (α_1, α_2) that we need to discuss in order to solve for the optimal portfolio. As can be seen from $u_2(1+\alpha_1+\alpha_2)$, the investor is always better off with $\alpha_1+\alpha_2 \geq 1$. This reduces the number of cases for analysis to 4.

We look at local maximum for each of the 4 possible cases.

1. $2\alpha_1+3\alpha_2 \leq -1, \alpha_1+\alpha_2 \geq 1$

$$E(u_2) = \frac{1}{6} * \frac{1}{3} + \frac{1}{2} * \frac{1}{3} + \frac{1}{3}(2\alpha_1+3\alpha_2).$$

The expected utility is clearly maximized at $2\alpha_1+3\alpha_2 = -1$, and the maximum utility in this semi-plan is $E(u_2) = \frac{1}{6} * \frac{1}{3} + \frac{1}{2} * \frac{1}{3} - \frac{1}{3} = -\frac{1}{9}$.

2. $-1 \leq 2\alpha_1+3\alpha_2 \leq 0, \alpha_1+\alpha_2 \geq 1$

$$\begin{aligned} E(u_2) &= \frac{1}{6}(-2\alpha_1-3\alpha_2) + \frac{1}{2} * \frac{1}{3} + \frac{1}{3}(2\alpha_1+3\alpha_2) \\ &= -\frac{1}{6}(2\alpha_1+3\alpha_2) + \frac{1}{6}. \end{aligned}$$

It follows that the expected utility is maximized at $2\alpha_1+3\alpha_2 = 0$, and the maximum utility achieved in the semi-plan is $E(u_2) = -\frac{1}{6} * 0 + \frac{1}{6} = \frac{1}{6}$.

3. $0 \leq 2\alpha_1 + 3\alpha_2 \leq 1, \alpha_1 + \alpha_2 \geq 1$

$$\begin{aligned} E(u_2) &= \frac{1}{6}(-2\alpha_1 - 3\alpha_2) + \frac{1}{2} * \frac{1}{3} + \frac{1}{3} * \frac{1}{3}(2\alpha_1 + 3\alpha_2) \\ &= -\frac{1}{18}(2\alpha_1 + 3\alpha_2) + \frac{1}{6}. \end{aligned}$$

the maximum is clearly achieved at $2\alpha_1 + 3\alpha_2 = 0$, and the maximum utility achieved in the semi-

plan is $E(u_2) = -\frac{1}{18} * 0 + \frac{1}{6} = \frac{1}{6}$.

4. $1 \leq 2\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2 \geq 1$

$$\begin{aligned} E(u_2) &= \frac{1}{6}(-2\alpha_1 - 3\alpha_2) + \frac{1}{2} * \frac{1}{3} + \frac{1}{3} * \frac{1}{3} \\ &= -\frac{1}{6}(2\alpha_1 + 3\alpha_2) + \frac{1}{6} + \frac{1}{9}. \end{aligned}$$

Since $-2\alpha_1 - 3\alpha_2 \leq -1$, the maximum is obtained at $2\alpha_1 + 3\alpha_2 = 1$, and the maximum utility in

this semi-plan is $E(u_2) = -\frac{1}{6} + \frac{1}{6} + \frac{1}{9} = \frac{1}{9}$.

The global maximum is then the set $\{(\alpha_1, \alpha_2) / 2\alpha_1 + 3\alpha_2 = 0, \alpha_1 + \alpha_2 \geq 1\}$, in which the maximum

utility level achieved is $\frac{1}{6}$.

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Table I
Joint Density Function of Example 3

This table presents the joint density function $f(x_1, x_2)$ of a quadrant dependent distribution. Quadrant dependence describes how two random variables behave together when they are simultaneously small or large. Here we observe that the two assets are positively quadrant dependent.

	x_2		
x_1	-3	1	3
-2	1/6	0	0
1	0	1/2	0
2	0	0	1/3

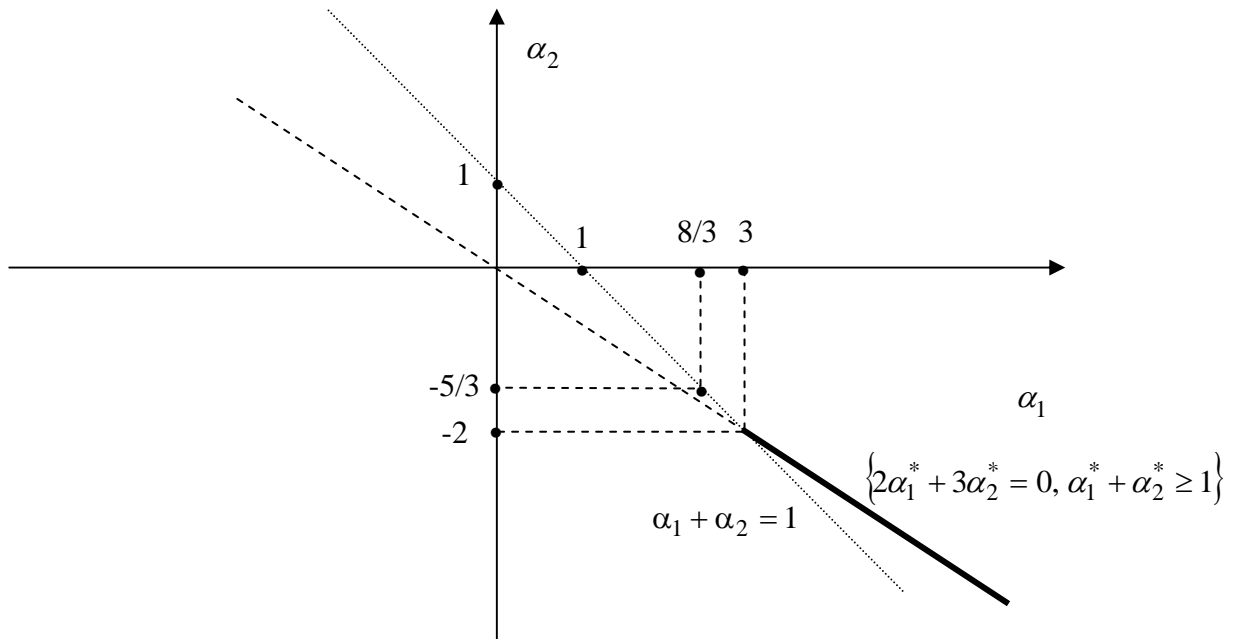


Figure 1. Optional solutions to Example 3. This figure depicts the optimal solution of investor u_1 at $(8/3, -5/3)$ and that of investor u_2 corresponding to the semi-line $2\alpha_1^* + 3\alpha_2^* = 0$ and $\alpha_1^* + \alpha_2^* \geq 1$ when the data are from Table I. The joint distribution of this example does not yield a separating solution since the point $(8/3, -5/3)$ is not on the semi-line.