

PARTIAL INFORMATION, DEFAULT HAZARD PROCESS, AND DEFAULT-RISKY BONDS*

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Abstract

This paper studies in some examples the role of information in a default-risk framework. We examine three types of information for a firm's unlevered asset value to the secondary bond market: the classical case of continuous and perfect information, observation of past and contemporaneous asset values at selected discrete times, and observation of contemporaneous asset value at discrete times. The third information filtration is contained in the second, which in turn, is contained in the first. We investigate the changes of the distributional properties of the default time and the properties of bond prices and credit spreads with the reductions of the information sets. Consistently with the observed market prices, model bond prices with partial information have surprise jumps prior to default. Credit spreads for very short times to maturities are increasing with the reductions of the information sets. High-yield bonds with the two types of incomplete information have downward sloping term structures of credit spreads. For firms with good credit qualities, increases of the observation lags lead to upward shifts in the term structures of credit spreads. The two information constrained models admit reduced form representations, in which the time of default is a totally inaccessible time with default arrival intensities, but it is better avoiding the intensity approach to valuation since the hazard process approach is more efficient.

Key words: credit risk, information, hazard process of a default time, intensity, default-risky bonds

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1 Introduction

The structural or firm-value models of default are based on specifications of the firm's unlevered asset value process. In the first models of this type (Black and Scholes [2]; Merton [29]), default can take place only at the maturity date of the debt if asset value is lower than debt face value. This is in contrast to the actual times of defaults on corporate debt to the extent that firms frequently default prior to their debts' maturities. To obtain a more realistic model for the default time, Black and Cox [3] reformulated the valuation problem as a first passage time problem of asset value to a deterministic boundary. In this model, a poor performance is captured by a low value of the firm's productive assets relative to its stipulated debt payments and default occurs when asset value falls to some exogenously specified barrier, referred to as the lower reorganization boundary.

In the standard first passage models, based on diffusions, the time of default is a predictable stopping time in the filtration generated by asset value and computations of its distribution are difficult, except in the case of a geometric Brownian motion. Zhou [32] puts forth a jump-diffusion process for asset value, in which default can come by surprise. Even though these models yield more realistic shapes of the term structure of credit spreads, as pointed out by this author, explicit solutions for first passage times in such settings are not known and the price of a defaultable bond is obtained by numerical methods. The standard geometric Brownian models are based on the assumption of perfect and continuous observation of the firm's asset value. While there have been many generalizations and extensions of the basic first passage approach, such as to stochastic interest rates (Longstaff and Schwartz [27]; Rutkowski [31]) and endogenous bankruptcy (Leland [24]; Leland and Toft [25]; Leland [26]), the effects of incomplete information have been considered only recently.

An alternative, so-called reduced form approach to the valuation of default-sensitive contingent claims (see, e.g., [17], [9], [23] and [11]) does not look at the structure of a firm's asset and liabilities directly, but assumes that the time of default τ is a totally inaccessible stopping time. The method is the compensator of the default process. When this compensator is absolutely continuous with respect to Lebesgue measure, its derivative is called the intensity of default. The "generalized" intensity is Λ_t , where $\mathbb{1}_{\tau \leq t} - \Lambda_{t \wedge \tau}$ is a martingale, in the case where Λ is not absolutely continuous w.r.t. Lebesgue measure. Consequently, the reduced form models relate the credit spread of a defaultable bond to the intensity of default. Intensity is usually taken to be a function of state variables that are in the information set of the market and are relevant for predicting the likelihood of default of the issuer. It can be also viewed as the instantaneous default probability. It is known from survival analysis that the survival probability (in the present context, the probability of a

bond surviving without default) is equal to the exponential of the negative of the hazard process. While the hazard process and the compensator are equal in the Cox processes setting (see, e.g., Lando [23]), in general, they may be different.

The first systematic study of the consequences of the incomplete information in the structural credit risk models is Duffie and Lando [10]. These authors noticed that frequently the standard assumption of continuous observation of asset value in the first-passage models is not satisfied in practice. While bond investors in a publicly traded firm can estimate the level of its assets by observing the prices of its equity and debt, the bondholders in a privately held firm have to rely on periodic accounting reports received at discrete dates. Therefore, the information flows to bondholders in such a firm can be more realistically modeled by the filtration generated by discrete observations of asset value. As noted by Duffie and Lando, for bondholders having such information, the default time is a totally inaccessible time and admits intensity. In this way, the authors reconcile the structural and reduced form approaches to the valuation of defaultable bonds and obtain higher and more realistic short-term credit spreads.

Kusuoka [22] and Nakagawa [30] investigate the cases of continuous but imperfect observation of asset value. These authors assume observation of smooth functions of asset value and solve the related filtering problems. With such information, the default time admits a default arrival intensity of the same form as in [10]. The paper of Cetin et al. [5] examines another case of continuous but incomplete information where default occurs when the firm's cash balances are negative for some pre-specified period and hit a default threshold level after that. These authors assume that the managers observe continuously the cash balances of the firm, while the market has a continuous subfiltration of the continuous manager's information filtration. This subfiltration consists of information on whether the firm is in financial distress, the duration of the distress and whether the default threshold has been reached. CreditMetrics (Gupton et al. [15]) and Giesecke [14] approach the incomplete information problem from another perspective. In these models, bond investors have incomplete information about the default-triggering barrier instead of the asset process. Actually, the first document assumes multiple uncertain barriers for the different rating classes and uses information from the rating transition probability matrix to solve for the barriers. Both studies provide solutions for the joint default probability and default correlation. Giesecke [12] treats the case of incomplete information for both the default barrier and the asset process. However, in the incomplete asset information case, he assumes no asset information at all.

This paper focuses on the evaluation of conditional expectations with respect to three different

types of asset value information in a diffusion setting. Our view is toward examining how the distributional properties of the default time and bond prices change with the changes of the information sets. We split the information of the secondary bond market as information generated by asset value and observations of the default time. We assume that the information generated by observations of asset value by the market is a strict subfiltration of the filtration generated by continuous and perfect observation of these assets. Following the hazard process approach (Elliott et al. [11]), the conditional law of the default time is the main tool to value contingent claims in the case where the default time is observed when it occurs. This valuation approach is more general than the intensity based valuation, to the extent that it does not require the knowledge of the compensator of the default process and the computations are independent of the properties of the compensator. The default probabilities conditional on asset information are important since they can be used as integrators when valuing contingent claims with non-zero recovery.

We examine the full filtration generated by asset value and two different asset process subfiltrations. First, we recall the classical case, in which asset value is continuously and perfectly observed and the information is represented by a Brownian filtration. We refer to this case as full information, and we note that the time of default is a stopping time in this filtration. Second, we examine the case when bondholders observe the contemporaneous and all past asset values on a sequence of selected discrete times, which represent a subfiltration of the full asset filtration. The main investors in defaultable bonds are investment funds. It is important to realize that a specific default-risky bond represents a small fraction of the portfolio of an investment fund. Since the investment funds have multi-asset portfolios, the investment fund managers do not have the ability to gather credit information continuously or to perform a detailed credit analysis for each individual name in their portfolios. It is costly to become fully and continuously informed about the value of the assets of a single issuer. If all the claims against the firm's assets are publicly traded, then asset value is observed since the value of the firm's assets is equal to that of its liabilities. The problem is that firms frequently have multiple liabilities, some of which are not very liquid. For example, a firm may have multiple classes of common and preferred stocks. Also, the defaultable bonds of small firms are not traded on organized exchanges and obtaining continuous prices of them may be difficult. In many cases, investment fund managers receive price data on some of the firm's contingent claims with an observation lag. So, obtaining continuous observations of asset value is difficult and costly. In practice, investment fund managers gather asset value information and reevaluate their bonds at selected discrete times of release of important information about the credit quality of the issuer and the second type of information models such information flows.

Third, as in Duffie and Lando [10], we examine the case when the secondary bond market observes only the contemporaneous asset values at selected discrete times. However, we assume perfect observation of asset value at these times instead of noisy observations as do these authors.

The two partial information subfiltrations are jump-discontinuous, incorporating surprises at the observation dates. The time of default changes from a predictable stopping time with full information to a totally inaccessible time with partial information. We obtain the semimartingale representations of the conditional default probabilities in the two subfiltrations. It turns out that their martingale components are pure discontinuous processes. In line with [10], we show that structural models with realistic informational assumptions (incomplete, discretely arriving information) for the secondary bond market admit reduced form representations. However, we have two specific real-world examples where the hazard process and generalized intensity (see, e.g., Jeanblanc and Rutkowski [18], [19]) are different. The default time admits risk-neutral default arrival intensities, but it seems better avoiding the intensity-based approach to valuation since working with the conditional hazard processes is direct and much simpler.

Duffie and Lando [10] obtain valuation formulas for defaultable bonds by numerical methods. Giesecke [12] provides analytical solutions for zero-recovery bonds only in the polar cases with no asset information and full information. We obtain analytical solutions for the prices of bonds with zero recovery in the two partial information subfiltrations. For the first type of incomplete information, we also compute the price of a bond with non-zero recovery, paid at the time when asset value hits the default barrier. While the bond prices with full information are continuous, the discrete information arrivals induce surprise jumps.

For very short-term maturities, the information reductions result in higher credit spreads of the defaultable bonds that can match the size of credit spreads on corporate bonds. In contrast to the upward-sloping term structures of credit spreads with full information, with incomplete information the term structures of credit spreads are downward sloping for speculative grade issuers (i.e., such with low observed last asset value). With partial asset information only, the spread term structure is downward sloping even for issuers with a good credit quality (a high last observed asset value).

For firms with good credit qualities, increases of the observation lags lead to upward shifts in the term structures of credit spreads, suggesting that they should release more information about their asset values and keep the secondary bond market informed if they want to lower the cost of their debt financing. Moreover, for the same sequence of observation times and asset value processes, a firm with the first type of incomplete asset information is more transparent to its bondholders and would have a lower credit spread on its debt than a firm with the second type

of information. Assuming the same default barrier for equityholders, we examine in detail the case when the two asset subfiltrations of bondholders are enlarged with observations of whether equityholders have liquidated the firm. In these expanded information sets, credit spreads with both types of incomplete information are the same. However, this will no longer be true in the general case, in which the default barriers of the holders of the different contingent claims are different.

The rest of the paper is organized as follows. In Section 2, we specify a standard first passage model for the default event assuming fractional recovery of par for the defaultable bonds. Section 3 defines the default process and examines the properties of its compensator. The hazard process approach to the valuation is also introduced. Section 4 introduces the first type of incomplete information and obtains analytical solutions for the conditional default probability, hazard process and defaultable bond prices. Section 5 investigates the second type of incomplete information and its consequences on the conditional law of default time and bond prices. Section 6 provides analysis of credit spreads. In Section 7, we compare the valuation of defaultable bonds by the hazard process and the intensity approaches. Section 8 contains concluding remarks.

2 Full information: continuous observations of asset value

In the first-passage models, it is usually assumed, mainly for computational issues, that the unlevered asset value V of a firm follows the geometric Brownian motion process

$$dV_t = V_t((\mu - \delta)dt + \sigma dW_t^*), \quad (1)$$

where μ and σ , (with $\sigma > 0$) are the drift rate and the percentage volatility of asset value, δ is the rate of payments to the holders of claims on the firm's assets and W^* is a Brownian motion under the historical probability P^* , constructed on a probability space $(\Omega, \mathbf{F}^V, P^*)$.

In the classical models such as Black and Cox [3], the information flows of the bondholders are modeled by filtrations generated by continuous observations of asset value. In particular, we denote by \mathbf{F}^V the complete filtration generated by V , i.e., $\mathbf{F}^V = (\mathcal{F}_t^V, t \geq 0)$, where $\mathcal{F}_t^V = \sigma(V_s, s \leq t)$. Such information sets are plausible for the firm's managers, but not so for the secondary bond market. In the subsequent analysis, we assume that the firm managers are also the equityholders of the firm.

The first passage approach accounts for the possibility of liquidation by equityholders as well as for such provisions of corporate indentures as safety covenants and debt subordination. Safety

covenants provide bondholders with the rights to enforce default, reorganization and liquidation of a firm prior to default when its financial performance is poor.

In particular, suppose that default occurs when the firm's asset value falls below some threshold level α , which is lower than the initial firm value $V_0 = v$. Then, the time of default τ is given by

$$\tau = \inf\{t : V_t \leq \alpha\} = \inf\{t : X_t \leq a\} \quad (2)$$

with $X_t = \frac{1}{\sigma} \ln(V_t/v)$ and $a = \frac{1}{\sigma} \ln(\alpha/v)$. Default can be forced by either the bondholders or the equityholders.

Suppose that it exists a savings account paying a constant short term interest rate r . Hence, if one assumes that the firm's asset value is a tradable asset, the market is complete and arbitrage-free. In what follows, the computations are carried under the risk neutral probability measure P . We recall that under P

$$dV_t = V_t((r - \delta)dt + \sigma dW_t), \quad (3)$$

where $(W_t, t \geq 0)$ is a (P, \mathcal{F}_t^V) -Brownian motion. The solution of (3) is $V_t = ve^{\sigma(W_t + \nu t)}$, where $\nu = \frac{1}{\sigma}(r - \delta - \frac{\sigma^2}{2})$. We rewrite the solution in the more convenient form

$$V_t = ve^{\sigma X_t},$$

where $X_t = \nu t + W_t$ is a (P, \mathcal{F}_t^V) -Brownian motion with drift. It is important to notice that the filtration \mathbf{F}^V is equal to the filtrations \mathbf{F}^X and \mathbf{F}^W generated by X and W , respectively.

Figure 1 displays $(X_t, 0 \leq t \leq T)$ and the default boundary with the base values of the parameters that we shall use in this study

$$\sigma = 0.30; r = 0.04; \delta = 0.03; v = 100; \alpha = 80; T = 2. \quad (4)$$

We choose a very high default boundary for illustrative purposes, since, in the simulations, we want to obtain paths close to default.

We denote the probability that the process $(X_s = W_s + \nu s, s \geq 0)$ remains above the barrier z till time t by

$$\Phi(\nu, t, z) = P\left(\inf_{s \leq t} X_s > z\right). \quad (5)$$

It is important to notice that this probability depends on the drift rate ν . The reflection principle and elementary considerations lead to

$$\begin{aligned} \Phi(\nu, t, z) &= \mathcal{N}\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z} \mathcal{N}\left(\frac{z + \nu t}{\sqrt{t}}\right), & \text{for } z < 0, t > 0, \\ &= 0, & \text{for } z \geq 0, t \geq 0, \\ \Phi(\nu, 0, z) &= 1, & \text{for } z < 0. \end{aligned}$$

Note that Φ , being a probability, satisfies $0 \leq \Phi \leq 1$.

The event that the firm does not default up to the maturity T of the bond can be expressed as

$$\begin{aligned} \{\tau > T\} &= \left\{ \inf_{s \leq T} V_s > \alpha \right\} = \left\{ \inf_{s \leq t} V_s > \alpha \right\} \cap \left\{ \inf_{t < s \leq T} V_s > \alpha \right\} \\ &= \left\{ \inf_{s \leq t} X_s > a \right\} \cap \left\{ \inf_{t < s \leq T} X_s - X_t > a - X_t \right\} \\ &= \left\{ \inf_{s \leq t} X_s > a \right\} \cap \left\{ \inf_{s \leq T-t} \widehat{X}_{s-t} > a - X_t \right\}, \end{aligned}$$

where $\widehat{X} = (\widehat{X}_u = X_{t+u} - X_t, u \geq 0)$ is independent of \mathcal{F}_t^V . Because of the stationarity and the independence of the increments of the Brownian motion, the process $(\widehat{X}_u, u \geq 0)$ is a (P, \mathcal{F}_{t+u}^V) -Brownian motion with drift.

Suppose that the firm has issued defaultable bonds with unit face values that pay their face values at maturity if there is no default or the recovery rate otherwise. We assume that there are no bankruptcy costs, so that bondholders receive all asset value at default, i.e. α . Let D be the total face value of the firm's outstanding bonds. Then, in the case of default, the residual firm value is distributed to all the bondholders and each bond recovers $\frac{\alpha}{D}$. A defaultable bond with zero recovery pays the face value of the bond at the maturity date if there is no default and zero otherwise.

To facilitate the subsequent analysis, we introduce two approximations to the Black and Cox [3] valuation problem. First, we assume that the default-triggering barrier for asset value is constant and equals α . Black and Cox provide a valuation formula for a barrier of the form $\alpha e^{-\eta(T-t)}$. In fact, the case of an exponential barrier $f(t) = \alpha e^{\eta t}$, $\tau = \inf\{t : V_t \leq f(t)\}$ reduces to the constant case up to a change of parameters

$$\tau = \inf\{t : x e^{\sigma X_t} \leq \alpha e^{\eta t}\} = \inf\{t : y e^{\sigma Y_t} \leq \alpha\}$$

with $Y_t = X_t - \frac{\eta}{\sigma}t$, $y = x$.

Second, in the formulation of Black and Cox, the default barrier would be α up to T and D at T . Following this approach, one would define the default time (see also Bielecki and Rutkowski [1]) as $\tau^* = \tau \wedge \tau^m$, where τ^m is the Merton's default time, i.e., $\tau^m = T$ if $V_T < D$ and $\tau^m = \infty$ otherwise. We work with the default time τ rather than τ^* .

With the above assumptions, when the information available to the market at time t is the σ -algebra \mathcal{F}_t^V , the default-risky bond price $B_d(t, T)$ is given by

$$\begin{aligned} B_d(t, T) &= E \left(e^{-r(T-t)} \mathbb{1}_{\tau > T} + \frac{\alpha}{D} e^{-r(\tau-t)} \mathbb{1}_{\tau \leq T} \middle| \mathcal{F}_t^V \right) \\ &= e^{-r(T-t)} P(\tau > T | \mathcal{F}_t^V) + \frac{\alpha}{D} E \left(e^{-r(\tau-t)} \mathbb{1}_{\tau \leq T} \middle| \mathcal{F}_t^V \right). \end{aligned}$$

The price of the defaultable bond is a sum of two components. We write $Z(t, T) := e^{-r(T-t)}P(\tau > T | \mathcal{F}_t^V)$ for the first term, which is the price of a defaultable bond with zero recovery. The second term is the value of the recovery, $R(t, T) := \frac{\alpha}{D}E(e^{-r(\tau-t)}\mathbb{1}_{\tau \leq T} | \mathcal{F}_t^V)$, in the case of default.¹

The conditional probability of survival to the maturity, $P(\tau > T | \mathcal{F}_t^V)$, can be expressed in terms of the probability of hitting the default barrier

$$\begin{aligned}
P(\tau > T | \mathcal{F}_t^V) &= P\left(\left\{\inf_{s \leq t} X_s > a\right\} \cap \left\{\inf_{t < s \leq T} V_s > \alpha\right\} \middle| \mathcal{F}_t^V\right) \\
&= \mathbb{1}_{t < \tau} P\left(\inf_{t \leq s \leq T} V_s > \alpha \middle| \mathcal{F}_t^V\right) \\
&= \mathbb{1}_{t < \tau} P\left(\inf_{s \leq T-t} \widehat{X}_s > a - X_t \middle| \mathcal{F}_t^V\right) \\
&= \mathbb{1}_{t < \tau} P\left(\inf_{s \leq T-t} \widehat{X}_s > a - x\right) \Big|_{x=X_t} \\
&= \mathbb{1}_{t < \tau} \Phi(\nu, T-t, a - X_t),
\end{aligned} \tag{6}$$

hence

$$Z(t, T) = e^{-r(T-t)} \mathbb{1}_{t < \tau} \Phi(\nu, T-t, a - X_t).$$

As for the recovery value, using absolute continuity relationship, we can write

$$R(t, T) = \frac{\alpha}{D} e^{rt} E^\nu(e^{-r\tau} \mathbb{1}_{\tau \leq T} | \mathcal{F}_t^V), \tag{7}$$

where $P^{(\nu)}$ is the probability (on the canonical space), such that the process X is a Brownian motion with drift ν .

Then,

$$\begin{aligned}
R(t, T) &= \frac{\alpha}{D} e^{rt} \frac{E^{(\gamma)}(L_\tau e^{-r\tau} \mathbb{1}_{\tau \leq T} | \mathcal{F}_t^V)}{L_t} \\
&= \frac{\alpha}{D} e^{rt} E^{(\gamma)}\left(e^{-r\tau} e^{(\nu-\gamma)(X_\tau - X_t) - \frac{\nu^2 - \gamma^2}{2}(\tau-t)} \mathbb{1}_{\tau \leq T} \middle| \mathcal{F}_t^V\right),
\end{aligned}$$

where $dP^{(\gamma)} = L_t^{-1} dP^{(\nu)} = \exp\left(-\gamma X_t - \frac{\gamma^2}{2}t\right) dP^{(0)}$, so that under $P^{(\gamma)}$ the process X is a Brownian motion with drift γ . We have $P^{(\gamma)}|_{\mathcal{F}_t^V} = L_t^{-1} P^{(\nu)}|_{\mathcal{F}_t^V}$, where

$$L_t = \exp\left((\nu - \gamma)X_t - \frac{\nu^2 - \gamma^2}{2}t\right).$$

If we choose γ such that $\gamma = \sqrt{2r + \nu^2}$, we obtain

$$\begin{aligned}
R(t, T) &= \frac{\alpha}{D} e^{(\nu-\gamma)(a-X_t)} E^{(\gamma)}(\mathbb{1}_{\tau \leq T} | \mathcal{F}_t^V) \\
&= \frac{\alpha}{D} e^{(\nu-\gamma)(a-X_t)} \left(1 - E^{(\gamma)}(\mathbb{1}_{T < \tau} | \mathcal{F}_t^V)\right) \\
&= \frac{\alpha}{D} e^{(\nu-\gamma)(a-X_t)} \left(1 - \mathbb{1}_{t < \tau} \Phi(\gamma, T-t, a - X_t)\right).
\end{aligned} \tag{8}$$

¹If the recovery is paid at the maturity date, the computation is simpler.

Finally, the price of the defaultable bond is

$$\begin{aligned}
B_d(t, T) &= e^{-r(T-t)} \mathbb{1}_{t < \tau} \Phi(\nu, T-t, a - X_t) \\
&+ \frac{\alpha}{D} e^{(\nu-\gamma)(a-X_t)} \left(1 - \mathbb{1}_{t < \tau} \Phi(\gamma, T-t, a - X_t) \right).
\end{aligned} \tag{9}$$

Equation (9) gives the price of a defaultable bond with fractional recovery (α/D) of par as opposed to the recovery payoff in the original Black and Cox [3] model. In the numerical examples in [3], fractional recovery of the net present value of par is assumed, that is, $(\alpha/D)e^{-r(T-t)}$. Moreover, we assumed that the total value of the bonds at the maturity date if the firm is not reorganized before is $\min(\alpha, D)$, while in [3], it is $\min(V_T, D)$.

As a check, for $r = 0$, the term on the right-hand-side of (9) reduces to $\mathbb{1}_{t < \tau} \Phi(\nu, T-t, a - X_t)(1 - \frac{\alpha}{D}) + \frac{\alpha}{D}$ as expected from the definition. Note that the time of default τ is a predictable stopping time in the filtration \mathbf{F}^V and the conditional default probability in this filtration is simply given by $F_t = P(\tau \leq t | \mathcal{F}_t^V) = \mathbb{1}_{\tau \leq t}$. It is obvious that, in this case, the price of the defaultable bond does not have surprise jumps. In the case of default, bond price converges continuously to its recovery value of $\frac{\alpha}{D}$ as the Brownian motion X approaches the barrier a . This is inconsistent with the empirical regularities, since the prices of defaultable bonds have multiple jumps prior to the time of default. Also, credit spreads generated by the model are too low when compared to the actual market spreads. In fact, the short term credit spreads are close to zero and, for short maturities, the term structure of credit spreads is upwardly sloping, which is in contrast to the market credit spreads. In the following section we examine bond prices with two types of partial information.

3 Default process, its compensator and hazard process

Let $\mathbf{F}^i = (\mathcal{F}_t^i, t \geq 0)$ be some reference filtration. We denote the conditional probability of default in this filtration by $F_t^i = P(\tau \leq t | \mathcal{F}_t^i)$. We write $\mathcal{G}_t^i = \mathcal{F}_t^i \vee \sigma(\tau \wedge t)$ for the σ -algebra enlarged with observations of the default event and $\mathbf{G}^i = (\mathcal{G}_t^i, t \geq 0)$ for the enlarged filtration. We also write $\Gamma_t^i = -\ln(1 - F_t^i)$ for the \mathbf{F}^i -hazard process, where we set $\Gamma_t^i = +\infty$ for $F_t^i = 1$.

The hazard process approach (Elliott et al. [11]; Jeanblanc and Rutkowski [19]) states that if X is some integrable, \mathcal{F}_T^i -measurable random variable, then

$$E[X \mathbb{1}_{T < \tau} | \mathcal{G}_t^i] = \mathbb{1}_{t < \tau} E \left[X e^{\Gamma_t^i - \Gamma_T^i} \middle| \mathcal{F}_t^i \right]. \tag{10}$$

Thus, the hazard process approach provides a method for evaluation of conditional expectations with respect to \mathbf{G}^i . We emphasize that the formula (10) holds true even if $(\Gamma_t^i, t \geq 0)$ does not

enjoy the increasing property. In particular, for $X = 1$, that is, the case of a defaultable bond, we are back to

$$P(T < \tau | \mathcal{G}_t^i) = \mathbb{1}_{t < \tau} \frac{P(T < \tau | \mathcal{F}_t^i)}{P(t < \tau | \mathcal{F}_t^i)}.$$

The hypothesis (H) (Jeanblanc and Rutkowski [19]) is that every square integrable martingale in the filtration \mathbf{F}^i is invariant under the enlargement of filtration with observations of default time, that is, it is also a \mathbf{G}^i -square integrable martingale.

In what follows, we will write $D_t := \mathbb{1}_{\tau \leq t}$ for the default process and $\mathbf{D} = (D_t, t \geq 0)$ for its natural filtration, where $\mathcal{D}_t = \sigma(D_u, u \leq t) = \sigma(\tau \wedge t)$. In the case where F^i is increasing and continuous, the process

$$D_t - \Gamma_{t \wedge \tau}^i = D_t - \int_0^{t \wedge \tau} \frac{dF_s^i}{1 - F_s^i}$$

is a martingale.

If F^i is increasing (this is called condition (G) in Jeanblanc and Rutkowski [19]), the process

$$D_t - \int_0^{t \wedge \tau} \frac{dF_s^i}{1 - F_{s-}^i}$$

is a martingale and $\Lambda_{t \wedge \tau}^i = \int_0^{t \wedge \tau} \frac{dF_s^i}{1 - F_{s-}^i}$ is an increasing process, and Λ_t^i is called the generalized intensity.

In the general case, where the submartingale F^i admits the decomposition $F^i = Z^i + A^i$ where A^i is a predictable increasing process and Z^i is an \mathbf{F}^i -martingale, the process

$$D_t - \int_0^{t \wedge \tau} \frac{dA_s^i}{1 - F_{s-}^i} \tag{11}$$

is a martingale, and $\int_0^t \frac{dA_s^i}{1 - F_{s-}^i}$ is the generalized intensity, which is different from the hazard process Γ^i .

Lemma 1 *The process $(\mathbb{1}_{\tau > t} e^{\Gamma_t^i})_{t \geq 0}$ is a \mathbf{G}^i -martingale.*

PROOF: For $s \geq t$,

$$\begin{aligned} E \left[\mathbb{1}_{\tau > s} e^{\Gamma_s^i} | \mathcal{G}_t^i \right] &= \mathbb{1}_{\tau > t} E \left[e^{\Gamma_s^i} e^{\Gamma_t^i - \Gamma_s^i} | \mathcal{F}_t^i \right] \\ &= \mathbb{1}_{\tau > t} e^{\Gamma_t^i}. \end{aligned}$$

△

4 First type of incomplete information: full observation at discrete times

The previous section shows that the asset value process is a necessary input in the valuation formula. In this and the following section, we retain the geometric Brownian motion process for the firm's asset value, but change the information sets available to the secondary bond market. We address the important empirical regularity that the managers of the bond investment funds, the main investors in defaultable bonds, are not completely informed about the financial status of a single firm to the extent that its asset value is not traded and is not perfectly and continuously observable. In such cases it has to be estimated, but in some cases, it is costly to do so continuously.

In the case of a publicly traded firm, the most natural way to estimate asset value is by looking at the prices of the contingent claims against its assets. However, we assume that the investment fund managers have observation lags for the prices of some of its contingent claims. Such a situation may arise if some of the contingent claims of the firm are less liquid and are traded over-the-counter. This is typically the case with small and medium-size firms. Then, obtaining the prices of all the contingent claims of the firm continuously and computing its asset value would use cognitive resources of investment fund managers, who have to manage multi-asset portfolios and the defaultable bond of the specific issuer is only a small fraction of them. For these reasons, the investment fund managers usually gather asset value information only at selected discrete times of announcement of important information about the credit quality of the issuer.

To address such a situation, we assume that the secondary bond market has access only to a subfiltration of the full-information filtration \mathbf{F}^V . More specifically, we assume that the bondholders obtain all the information about contemporaneous and past values of the firm's assets at the dates

$$\mathbb{T} = \{t_1, t_2, t_3, \dots\},$$

where $(t_i, 1 \leq i)$ is an increasing sequence of observation times.

We denote by $\mathbf{H} = (\mathcal{H}_t, t \geq 0)$ the filtration generated by the observations of contemporaneous and past V at times t_1, \dots, t_n with $t_n \leq t < t_{n+1}$, that is,

$$\begin{aligned} \mathcal{H}_t &= \{\emptyset, \Omega\} && \text{for } t < t_1, \\ \mathcal{H}_t &= \mathcal{F}_{t_1}^V = \sigma(V_s, s \leq t_1) && \text{for } t_1 \leq t < t_2, \\ \mathcal{H}_t &= \mathcal{F}_{t_n}^V = \sigma(V_s, s \leq t_n) && \text{for } t_n \leq t < t_{n+1}. \end{aligned}$$

Our aim is to characterize the conditional distributional properties of τ when the available information is \mathbf{H} .

The filtration \mathbf{H} is a subfiltration of \mathbf{F}^V (i.e., $\mathbf{H} \subset \mathbf{F}^V$). It is "constant" between the observation dates. At the observation date t_i , the σ -algebra \mathcal{H}_t is enlarged by $\sigma(V_s, t_{i-1} \leq s < t_i)$, that is, $\mathcal{H}_{t_i} = \mathcal{H}_{t_{i-1}} \vee \sigma(V_s, t_{i-1} \leq s < t_i)$. We write $F_t^1 := P(\tau \leq t | \mathcal{H}_t)$ for the \mathbf{H} -conditional default probability, where the superscript 1 stands for the first type of information. We denote by $\Gamma_t^1 := -\ln(1 - F_t^1)$ the \mathbf{H} -hazard process of τ . In this section and the following one, we shall write $\Phi(t, a)$ for $\Phi(\nu, t, a)$.

4.1 First interval

In the case $t < t_1$, since the information is the trivial one, the conditional default probability F_t^1 is deterministic and is equal to the distribution function of τ . Using the fact that $a < 0$, we obtain

$$F_t^1 = P(\tau \leq t) = P\left(\inf_{s \leq t} X_s \leq a\right) = 1 - \Phi(t, a)$$

and F^1 is continuous on $[0, t_1[$.

4.2 Second interval

In the case $t_1 \leq t < t_2$, we have

$$\begin{aligned} F_t^1 &= P(\tau \leq t | \mathcal{H}_{t_1}) = 1 - P(\tau > t | \mathcal{H}_{t_1}) \\ &= 1 - P\left(\inf_{s < t} X_s > a \middle| \mathcal{F}_{t_1}^V\right) = 1 - \mathbb{1}_{\{\inf_{s < t_1} X_s > a\}} P\left(\inf_{t_1 \leq s < t} X_s > a \middle| \mathcal{F}_{t_1}^V\right). \end{aligned} \quad (12)$$

By using (6) and (12), we obtain

$$F_t^1 = 1 - \mathbb{1}_{\tau > t_1} \Phi(t - t_1, a - X_{t_1}).$$

Note that $F_{t_1}^1 = 0$ on the set $\{\tau > t\}$.

We denote by ΔF_s^1 the jump of F^1 at s , i.e., $\Delta F_s^1 = F_s^1 - F_{s-}^1$. The process F^1 is continuous, increasing and deterministic on $[t_1, t_2[$; on the set $\{\tau \leq t_1\}$, the jump of F^1 at time t_1 is positive with size

$$\Delta F_{t_1}^1 = 1 - (1 - \Phi(t_1, a)) = \Phi(t_1, a),$$

while on the set $\{\tau > t_1\}$ it is a negative jump equal to

$$\begin{aligned} \Delta F_{t_1}^1 &= 1 - \Phi(0, a - X_{t_1}) - (1 - \Phi(t_1, a)) \\ &= \Phi(t_1, a) - 1. \end{aligned}$$

In the above computation, we have used that, on $\{\tau > t_1\}$, the inequality $a < X_{t_1}$ holds, hence $\Phi(0, a - X_{t_1}) = 1$. Note that if $F_t^1 = 1$, then $F_{t+s}^1 = 1 \quad \forall s > 0$.

4.3 General case

We easily extend the previous result to the following observation times. For $t_i \leq t < t_{i+1}$, we can write

$$\begin{aligned} F_t^1 &= P(\tau \leq t | \mathcal{H}_t) = 1 - P(\tau > t | \mathcal{F}_{t_i}^V) \\ &= 1 - P\left(\inf_{s < t} X_s > a \mid \mathcal{F}_{t_i}^V\right) = 1 - \mathbb{1}_{\tau > t_i} P\left(\inf_{t_i \leq s < t} X_s > a \mid \mathcal{F}_{t_i}^V\right) \\ &= 1 - \mathbb{1}_{\tau > t_i} \Phi(t - t_i, a - X_{t_i}). \end{aligned}$$

If there is default between two observations, the process F^1 jumps to one on the next observation date. The \mathbf{H} -hazard process is given by

$$\Gamma_t^1 = -\ln(\mathbb{1}_{\tau \geq t_i} \Phi(t - t_i, a - X_{t_i})),$$

where we allow the value of $+\infty$ for Γ^1 . Prior to default, there are jumps in F^1 and Γ^1 to zero at the times of observation of the firm's assets.

The process F^1 is continuous and increasing on $[t_i, t_{i+1}[$; on the set $\{\tau \leq t_i\}$, its jump at time t_i is positive

$$\begin{aligned} \Delta F_{t_i}^1 &= 1 - (1 - \mathbb{1}_{\tau > t_{i-1}} \Phi(t_i - t_{i-1}, a - X_{t_{i-1}})) \\ &= \mathbb{1}_{\tau > t_{i-1}} \Phi(t_i - t_{i-1}, a - X_{t_{i-1}}), \end{aligned}$$

while on the set $\{\tau > t_i\}$ it is a negative jump equal to

$$\begin{aligned} \Delta F_{t_i}^1 &= 1 - \Phi(0, a - X_{t_i}) - (1 - \Phi(t_i - t_{i-1}, a - X_{t_{i-1}})) \\ &= \Phi(t_i - t_{i-1}, a - X_{t_{i-1}}) - 1. \end{aligned}$$

Note that the probability of the bond surviving without default is given by

$$\begin{aligned} P(\tau > T | \mathcal{H}_t) &= P\left(\inf_{s < T} X_s > a \mid \mathcal{F}_{t_i}^V\right) \\ &= \mathbb{1}_{\tau > t_i} P\left(\inf_{t_i < s < T} X_s > a \mid \mathcal{F}_{t_i}^V\right) \\ &= \mathbb{1}_{\tau > t_i} \Phi(T - t_i, a - X_{t_i}) \quad \text{for } t_i \leq t < T. \end{aligned} \tag{13}$$

The process F^1 jumps to either one or zero at each observation date and this is illustrated in the following two figures. Figure 2 displays the conditional default probability ($F_t^1, 0 \leq t \leq T$) and the hazard process ($\Gamma_t^1, 0 \leq t \leq T$) for the following sequence of observation dates: $t_1 = 0.5$, $t_2 = 1$ and $t_3 = 1.5$. We have simulated the probability with the typical values of the parameters and the

simulations resulted in a path with no default. These jumps are with stochastic sizes depending on V_{t_1}, V_{t_2} and V_{t_3} for t_2, t_3 and t_4 , respectively.

Figure 3 plots other simulation results for the same processes. In this example, default occurs at time $\tau = 0.624$. At the next observation date, $t_2 = 1$, the conditional default probability jumps to one and the hazard process jumps to $+\infty$.

Lemma 2 *The pure jump process ζ^1 defined by $\zeta_t^1 = \sum_{i, t_i \leq t} \Delta F_{t_i}^1$ is an \mathbf{H} -martingale.*

PROOF: Consider first the times $t_i \leq s < t \leq t_{i+1}$. In this case, $E(\zeta_t^1 | \mathcal{H}_s) = \zeta_s^1$ since $\zeta_t^1 = \zeta_s^1 = \zeta_{t_i}^1$ is \mathcal{H}_s -measurable.

It suffices to show that $E(\zeta_t^1 | \mathcal{H}_s) = \zeta_s^1$ for $t_i \leq s < t_{i+1} \leq t < t_{i+2}$. In this case $\zeta_t^1 = \zeta_{t_i}^1 + \Delta F_{t_{i+1}}^1$ and $\zeta_s^1 = \zeta_{t_i}^1$. We have

$$\begin{aligned}
E(\Delta F_{t_{i+1}}^1 | \mathcal{H}_s) &= E\left(\mathbb{1}_{\tau > t_i} \mathbb{1}_{\tau \leq t_{i+1}} \Phi(t_{i+1} - t_i, a - X_{t_i}) + \mathbb{1}_{\tau > t_{i+1}} (\Phi(t_{i+1} - t_i, a - X_{t_i}) - 1) | \mathcal{H}_s\right) \\
&= \mathbb{1}_{\tau > t_i} \Phi(t_{i+1} - t_i, a - X_{t_i}) - E(\mathbb{1}_{\tau > t_{i+1}} | \mathcal{H}_s) \\
&= \mathbb{1}_{\tau > t_i} \Phi(t_{i+1} - t_i, a - X_{t_i}) - E(\mathbb{1}_{\tau > t_{i+1}} | \mathcal{F}_{t_i}^V) \\
&= \mathbb{1}_{\tau > t_i} \Phi(t_{i+1} - t_i, a - X_{t_i}) - \mathbb{1}_{\tau > t_i} \Phi(t_{i+1} - t_i, a - X_{t_i}) \\
&= 0.
\end{aligned}$$

△

Accordingly, the Doob-Meyer decomposition of the submartingale $(F_t^1, t \geq 0)$ is

$$F_t^1 = \zeta_t^1 + (F_t^1 - \zeta_t^1),$$

where ζ^1 is an \mathbf{H} -martingale; $F^{1,c} := F^1 - \zeta^1$ is a continuous, hence predictable increasing process. Since F^1 is not increasing, the hypothesis (H) does not hold.

4.4 Implications for bond prices

4.4.1 Bond prices with observation of the time of default

In the literature, it is usually assumed that default is announced as it occurs and the default time is observed by the market. The question that arises naturally is how could bondholders with discrete information about asset value observe whether the default threshold has been reached. According to the US Bankruptcy Code, default and liquidation of the firm's assets can be done by the equityholders. In the subsequent analysis, we assume that the reorganization boundary for the equityholders is the same as for the bondholders and that the equityholders announce publicly their decision to liquidate the firm.

Figure 1: Path of the drifted Brownian motion ($X_t, 0 \leq t \leq T$) and the default boundary a

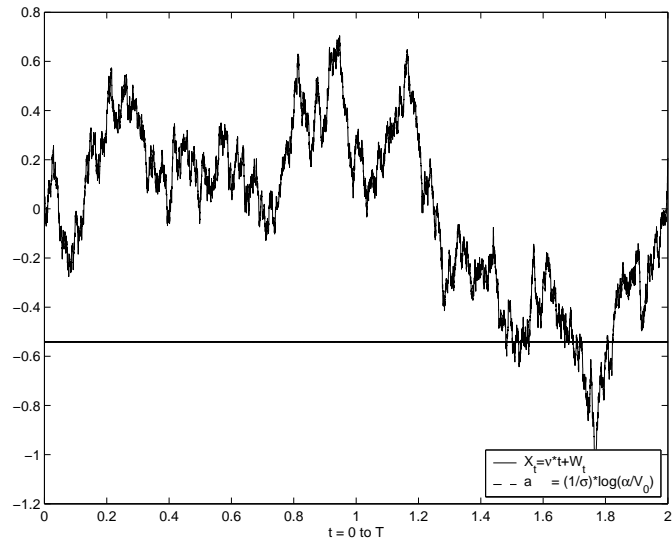


Figure 2: Default probability $F_t^1 = P(\tau < t | \mathcal{H}_t)$ and hazard process $\Gamma_t^1 = -\ln(1 - F_t^1)$ with information of the first type (path without default)

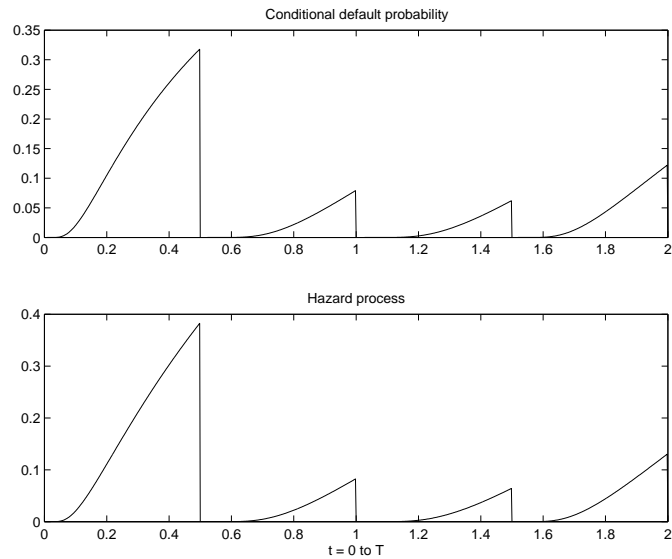


Figure 3: Default probability $F_t^1 = P(\tau < t | \mathcal{H}_t)$ and hazard process $\Gamma_t^1 = -\ln(1 - F_t^1)$ with information of the first type (path with default)

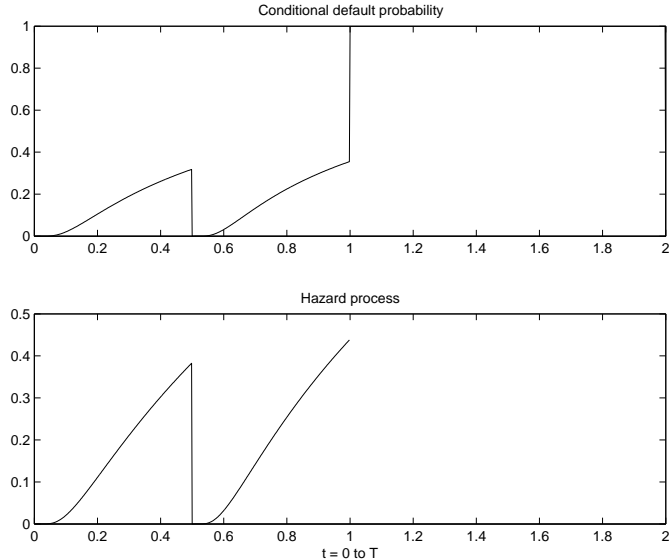
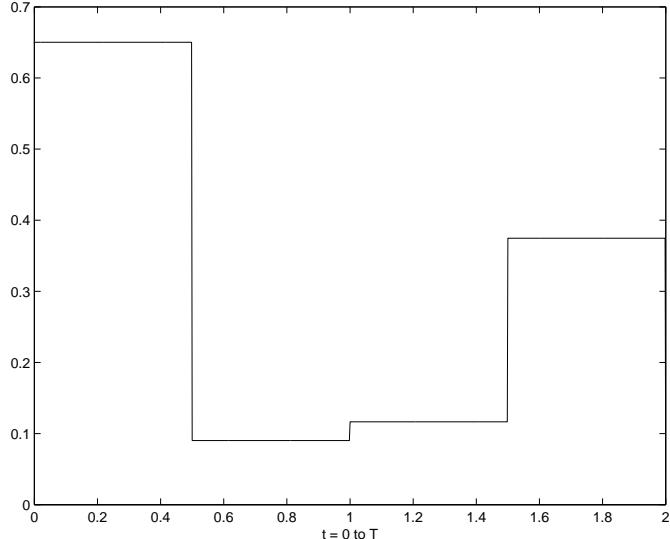


Figure 4: Probability of default prior to maturity with information of the first type $P(\tau < T | \mathcal{H}_t)$



The following theorem provides an analytical valuation formula for a defaultable bond with zero recovery when the flow of information is \mathbf{H} and observations of the default time.

Theorem 1 *When the information available to bondholders is $\mathcal{H}_t \vee \sigma(\tau \wedge t)$, the price of the zero-recovery defaultable bond is, for $t_i < t < t_{i+1}; t < T$,*

$$Z^1(t, T) = e^{-r(T-t)} \mathbb{1}_{t < \tau} \frac{\Phi(T - t_i, a - X_{t_i})}{\Phi(t - t_i, a - X_{t_i})}. \quad (14)$$

PROOF: The price of the zero-recovery defaultable bond is given by the discounted value of the conditional survival probability, i.e.,

$$Z^1(t, T) = e^{-r(T-t)} P(T < \tau | \mathcal{H}_t \vee \sigma(\tau \wedge t)) = e^{-r(T-t)} \mathbb{1}_{t < \tau} \frac{P(T < \tau | \mathcal{H}_t)}{P(t < \tau | \mathcal{H}_t)}, \quad (15)$$

where the second equality follows from the hazard process approach (10). To compute the fraction on the right-hand-side, we use the fact that

$$P(\tau > s | \mathcal{H}_t) = \mathbb{1}_{\tau > t_i} \Phi(s - t_i, a - X_{t_i}) \quad \text{for } s \geq t,$$

hence,

$$P(T < \tau | \mathcal{H}_t \vee \sigma(\tau \wedge t)) = \mathbb{1}_{t < \tau} \frac{\Phi(T - t_i, a - X_{t_i})}{\Phi(t - t_i, a - X_{t_i})}.$$

A substitution of this expression in (15) yields the result. △

The next theorem provides a formula for a defaultable bond with fractional recovery of par.

Theorem 2 *For $t_i < t < t_{i+1}$ and $t < T$, the price of the defaultable bond with fractional recovery paid at hit, when the information available on the market is $\mathcal{H}_t \vee \sigma(\tau \wedge t)$, is given by*

$$\begin{aligned} B_d^1(t, T) &= e^{-r(T-t)} \mathbb{1}_{t < \tau} \frac{\Phi(T - t_i, a - X_{t_i})}{\Phi(t - t_i, a - X_{t_i})} \\ &+ \frac{\alpha \mathbb{1}_{t < \tau} e^{r(t-t_i)} e^{(\nu-\gamma)(a-X_{t_i})}}{D \Phi(t - t_i, a - X_{t_i})} \{ \Phi(\gamma, t - t_i, a - X_{t_i}) - \Phi(\gamma, T - t_i, a - X_{t_i}) \}. \end{aligned} \quad (16)$$

PROOF: With information $\mathcal{H}_t \vee \sigma(\tau \wedge t)$, the value of a defaultable bond is

$$B_d^1(t, T) = Z^1(t, T) + R^1(t, T),$$

where $R^1(t, T)$ is the value of the rebate part. It is given by

$$R^1(t, T) = \frac{\alpha}{D} E \left(e^{-r(\tau-t)} \mathbb{1}_{t < \tau < T} | \mathcal{H}_t \vee \sigma(\tau \wedge t) \right) = \frac{\alpha}{D} \frac{\mathbb{1}_{t < \tau}}{P(t < \tau | \mathcal{H}_t)} E \left(e^{-r(\tau-t)} \mathbb{1}_{t < \tau < T} | \mathcal{H}_t \right).$$

Assume that $t_i < t < t_{i+1}$. From the Markov property,

$$E \left(e^{-r(\tau-t)} \mathbb{1}_{t < \tau < T} | \mathcal{H}_t \right) = E \left(e^{-r(\tau-t)} \mathbb{1}_{t < \tau < T} | \mathcal{F}_{t_i}^V \right). \quad (17)$$

One can write

$$E \left(e^{-r(\tau-t_i)} \mathbb{1}_{t_i < \tau < T} | \mathcal{H}_t \right) = \Psi(a - X_{t_i}, T - t_i),$$

where $\Psi(x, u) = E(e^{-r\tau} \mathbb{1}_{\tau < u})$ with $\tau = \inf\{s : \nu s + W_s \leq x\}$. It follows that

$$\begin{aligned} \mathbb{1}_{t < \tau} E \left(e^{-r(\tau-t)} \mathbb{1}_{t < \tau < T} | \mathcal{H}_t \right) &= \mathbb{1}_{t < \tau} E \left(e^{-r(\tau-t)} \mathbb{1}_{t < \tau < T} | \mathcal{F}_{t_i}^V \right) \\ &= \mathbb{1}_{t < \tau} \left[e^{r(t-t_i)} E \left(e^{-r(\tau-t_i)} \mathbb{1}_{t_i < \tau < T} | \mathcal{F}_{t_i}^V \right) \right. \\ &\quad \left. - e^{r(t-t_i)} E \left(e^{-r(\tau-t_i)} \mathbb{1}_{t_i < \tau < t} | \mathcal{F}_{t_i}^V \right) \right] \\ &= \mathbb{1}_{t < \tau} e^{r(t-t_i)} \left[\Psi(a - X_{t_i}, T - t_i) - \Psi(a - X_{t_i}, t - t_i) \right]. \end{aligned}$$

The computation of $\Psi(\cdot, \cdot)$ has been done in section 2. For $x = a - X_{t_i} < 0$,

$$\begin{aligned} \Psi(x, t) &= e^{(\nu-\gamma)x} \mathcal{N} \left(\frac{-\gamma t + x}{\sqrt{t}} \right) + e^{(\nu+\gamma)x} \mathcal{N} \left(\frac{\gamma t + x}{\sqrt{t}} \right) \\ &= e^{(\nu-\gamma)x} \left[1 - \mathcal{N} \left(\frac{\gamma t - x}{\sqrt{t}} \right) + e^{2\gamma x} \mathcal{N} \left(\frac{\gamma t + x}{\sqrt{t}} \right) \right] \\ &= e^{(\nu-\gamma)x} [1 - \Phi(\gamma, t, x)] \end{aligned} \tag{18}$$

with $\gamma = \sqrt{2r + \nu^2}$. Therefore,

$$\Psi(a - X_{t_i}, T - t_i) - \Psi(a - X_{t_i}, t - t_i) = e^{(\nu-\gamma)(a-X_{t_i})} \{ \Phi(\gamma, t - t_i, a - X_{t_i}) - \Phi(\gamma, T - t_i, a - X_{t_i}) \}.$$

Consequently, on the set $\{t < \tau\}$, the value of the rebate is given by

$$R^1(t, T) = \frac{\alpha \mathbb{1}_{t < \tau} e^{r(t-t_i)} e^{(\nu-\gamma)(a-X_{t_i})}}{D \Phi(t - t_i, a - X_{t_i})} \{ \Phi(\gamma, t - t_i, a - X_{t_i}) - \Phi(\gamma, T - t_i, a - X_{t_i}) \}.$$

A substitution in the price of defaultable bond leads to the result. \triangle

A comparison between $B_d(t, T)$ and $B_d^1(t, T)$ shows that the former depends on contemporaneous asset value (as indicated by the presence of X_t in (9)), while the latter depends on asset value at the last observation date (through X_{t_i} in (16)). Of course, both bond prices depend on whether the default-triggering barrier has been reached. However, it is clear that when the bondholders have access to \mathbf{F}^V , they use more recent information to value default-risky bonds. Also, $B_d^1(t, T)$ has multiple jumps to the $B_d(t, T)$ values at the observation dates prior to default. This is consistent with the empirically observed jumps in the prices of defaultable bonds.

4.4.2 Bond prices when \mathbf{H} is the only information

Theorem 3 For $t_i < t < t_{i+1}$ and $t < T$, the price of the defaultable bond with a maturity date T when the information available on the market is \mathcal{H}_t is given by

$$\begin{aligned}\tilde{B}_d^1(t, T) &= \mathbb{1}_{\tau > t_i} e^{-r(T-t)} \Phi(T - t_i, a - X_{t_i}) \\ &+ \mathbb{1}_{\tau > t_i} \frac{\alpha}{D} e^{rt} \left\{ \tilde{\Psi}(T, X_{t_i}) - \tilde{\Psi}(t, X_{t_i}) \right\}.\end{aligned}\quad (19)$$

PROOF: When bondholders have access only to the filtration \mathbf{H} , the price of the defaultable bond is

$$\begin{aligned}\tilde{B}_d^1(t, T) &= E \left(e^{-r(T-t)} \mathbb{1}_{\tau > T} + \frac{\alpha}{D} e^{-r(\tau-t)} \mathbb{1}_{\tau \leq T} \middle| \mathcal{H}_t \right) \\ &= e^{-r(T-t)} P(\tau > T | \mathcal{H}_t) + \frac{\alpha}{D} E \left(e^{-r(\tau-t)} \mathbb{1}_{\tau \leq T} \middle| \mathcal{H}_t \right) \\ &= \tilde{Z}^1(t, T) + \tilde{R}^1(t, T).\end{aligned}$$

The first term on the right-hand side is the value of a zero-recovery defaultable bond in the filtration \mathbf{H} , while the second term is the value of the rebate in the same filtration. The price of the zero recovery bond is the discounted value of the survival probability in (13)

$$\tilde{Z}^1(t, T) = \mathbb{1}_{\tau > t_i} e^{-r(T-t)} \Phi(T - t_i, a - X_{t_i}). \quad (20)$$

If \mathbf{H} is the only information, we have to assume that the recovery is paid at time t_{i+1} when $t_i < \tau < t_{i+1}$. Hence,

$$\begin{aligned}\tilde{R}^1(t, T) &= \frac{\alpha}{D} E \left(e^{-r(\tau-t)} \mathbb{1}_{t < \tau < T} \middle| \mathcal{H}_t \right) \\ &= \frac{\alpha}{D} e^{rt} \left\{ E \left(e^{-r\tau} \mathbb{1}_{\tau < T} \middle| \mathcal{H}_t \right) - E \left(e^{-r\tau} \mathbb{1}_{\tau < t} \middle| \mathcal{H}_t \right) \right\}.\end{aligned}$$

For $t_i < \tau < t_{i+1}$,

$$\begin{aligned}E \left(e^{-r\tau} \mathbb{1}_{\tau < T} \middle| \mathcal{H}_t \right) &= E \left(e^{-r\tau} \mathbb{1}_{\tau < T} \middle| \mathcal{F}_t^V \right) \\ &= E_{X_{t_i}} \left(e^{-r\tau} \mathbb{1}_{\tau < T} \right) = \tilde{\Psi}(T, X_{t_i}).\end{aligned}$$

The function $\tilde{\Psi}$ is defined by

$$\tilde{\Psi}(T, x) = E_x \left(e^{-r\tau} \mathbb{1}_{\tau < T} \right),$$

where $\tau = \inf\{t : X_{t-t_i} < a - x\}$.

△

Both the value of the zero recovery bond and the value of the rebate depend on the probability of survival to the maturity date, given in (13), which is equal to one minus the probability of

default prior to the maturity. Figure 4 displays the probability of default prior to the maturity. It is piecewise constant between two observations and jumps at the observation dates with stochastic sizes depending on asset values. These jumps induce jumps in the price of the defaultable bond.

5 Second type of partial information

In this section, we examine default probability and bond prices with a second type of incomplete information, which is a further restriction of that of the first type. As in Duffie and Lando [10], suppose that the bondholders observe asset value at selected discrete times. As pointed out by these authors, such a situation is relevant for a privately held firm, whose stockholders are not allowed to trade on the bond markets and whose bondholders obtain accounting information about its assets. The observation times may correspond to the dates of release of periodic accounting reports, such as balance sheets, profit and loss statements and cash flow statements by the firm. We write

$$\mathbb{T} = \{t_1, t_2, \dots, t_n\}$$

for the sequence of discrete observation times. In this case, the secondary bond markets receives information only about the contemporaneous asset value rather than the contemporaneous and past asset values as was the case examined in the previous section. It is important to note that we consider the case of perfect observations, while [10] examine the case where the log of asset value is observed in Gaussian noise.

We denote by $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$ the filtration generated by V at dates t_1, \dots, t_n . It follows that \mathcal{F}_t is trivial for $t < t_1$, i.e.,

$$\begin{aligned} \mathcal{F}_t &= \{\emptyset, \Omega\} && \text{for } t < t_1, \\ \mathcal{F}_t &= \mathcal{F}_{t_1} = \sigma(V_{t_1}) = \sigma(X_{t_1}) && \text{for } t_1 \leq t < t_2, \\ \mathcal{F}_t &= \mathcal{F}_{t_2} = \sigma(V_{t_1}, V_{t_2}) = \sigma(X_{t_1}, X_{t_2}) && \text{for } t_2 \leq t < t_3, \end{aligned}$$

and so on. At each observation date t_i , the filtration \mathbf{F} is enlarged with the observation of asset value at that date, that is, $\mathcal{F}_{t_i} = \mathcal{F}_{t_{i-1}} \vee \sigma(V_{t_i})$. Therefore, the bondholders have access to a subfiltration of the filtration \mathbf{F}^V (i.e., $\mathbf{F} \subset \mathbf{F}^V$). We denote the conditional default probability with respect to \mathbf{F} by $F_t^2 := P(\tau \leq t | \mathcal{F}_t)$, where the superscript 2 stands for the second type of information.

5.1 On $t < t_1$

As with the first type of information, before t_1 bondholders do not observe anything, and we have

$$F_t^2 = 1 - \Phi(t, a). \quad (21)$$

5.2 On $t_1 < t < t_2$

In this interval, the conditional default probability is given by

$$F_t^2 = P(\tau \leq t | X_{t_1}) = 1 - P(\tau > t | X_{t_1}), \quad (22)$$

where

$$\begin{aligned} P(\tau > t | X_{t_1}) &= P\left(\inf_{s < t} X_s > a | X_{t_1}\right) \\ &= P\left(\inf_{s < t_1} X_s > a, \inf_{t_1 \leq u < t} X_u > a | X_{t_1}\right) \\ &= E\left(\mathbb{1}_{\inf_{s < t_1} X_s > a} P\left(\inf_{t_1 \leq s < t} X_s > a | \mathcal{F}_{t_1}^V\right) | X_{t_1}\right) \end{aligned}$$

Using (6), we can evaluate the probability inside the expectation

$$P\left(\inf_{t_1 \leq s < t} X_s > a | \mathcal{F}_{t_1}^V\right) = \Phi(t - t_1, a - X_{t_1}).$$

Substituting above, we obtain

$$F_t^2 = 1 - \Phi(t - t_1, a - X_{t_1}) P\left(\inf_{s < t_1} X_s > a | X_{t_1}\right). \quad (23)$$

The term $P(\inf_{s < t_1} X_s > a | X_{t_1})$ corresponds to the probability of first passage time for a drifted Brownian bridge. In the following lemma, this term is computed by using the joint law of the infimum and the current position of a Brownian motion with drift. This is a classic result, we provide a demonstration to facilitate understanding.

Lemma 3 *Let $X_t = W_t + \nu t$ and $m_t^X = \inf_{s \leq t} X_s$. Then, for $y < 0$, $y < x$*

$$P(m_t^X \leq y | X_t = x) = \exp\left(-\frac{2}{t}y(y - x)\right). \quad (24)$$

PROOF: We know that, for $y \leq 0$, $y \leq x$,

$$P(X_t \geq x, m_t^X \leq y) = e^{2\nu y} \mathcal{N}\left(\frac{-x + 2y + \nu t}{\sqrt{t}}\right). \quad (25)$$

Taking the derivative with respect to x leads to

$$P(X_t \in dx, m_t^X \leq y) = \frac{1}{\sqrt{2\pi t}} e^{2\nu y} \exp\left(-\frac{1}{2}\left(\frac{-x + 2y + \nu t}{\sqrt{t}}\right)^2\right) dx.$$

By the definition of X , we have

$$P(X_t \in dx) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\left(\frac{x - \nu t}{\sqrt{t}}\right)^2\right) dx.$$

Since

$$P(m_t^X \leq y | X_t = x) = \frac{P(X_t \in dx, m_t^X \leq y)}{P(X_t \in dx)},$$

the equality follows from

$$\begin{aligned} 2\nu y - \frac{1}{2}\left(\frac{-x + 2y + \nu t}{\sqrt{t}}\right)^2 + \frac{1}{2}\left(\frac{x - \nu t}{\sqrt{t}}\right)^2 &= 2\nu y - \frac{1}{2t}(4y^2 + 4y(-x + \nu t)) \\ &= -\frac{2}{t}y(y - x). \end{aligned}$$

△

Note that the right-hand side of (24) does not depend on ν . This is, in fact, a consequence of Girsanov's theorem. Since $P(\inf_{s < t_1} X_s > a | X_{t_1}) = 1 - P(\inf_{s < t_1} X_s \leq a | X_{t_1})$, we can use (24) to evaluate (23).

For $X_{t_1} > a$, we obtain

$$F_t^2 = 1 - \Phi(t - t_1, a - X_{t_1}) \left[1 - \exp\left(-\frac{2}{t_1}a(a - X_{t_1})\right) \right]. \quad (26)$$

The case $X_{t_1} \leq a$ corresponds to default and, therefore, for $X_{t_1} \leq a$, $F_t^2 = 1$.

The process F^2 is continuous and increasing in $[t_1, t_2[$. When t approaches t_1 from above, for $X_{t_1} > a$, $F_{t_1^+}^2 = \exp\left[-\frac{2}{t_1}a(a - X_{t_1})\right]$, because $\lim_{t \rightarrow t_1^+} \Phi(t - t_1, a - X_{t_1}) = 1$.

For $X_{t_1} > a$, the jump of F^2 at t_1 is

$$\Delta F_{t_1}^2 = \exp\left[-\frac{2}{t_1}a(a - X_{t_1})\right] - 1 + \Phi(t_1, a).$$

For $X_{t_1} \leq a$, $\Phi(t - t_1, a - X_{t_1}) = 0$ by the definition of $\Phi(\cdot)$ and

$$\Delta F_{t_1}^2 = \Phi(t_1, a).$$

5.3 General observation times $t_i < t < t_{i+1} < T$, $i \geq 2$

For $t_i < t < t_{i+1}$,

$$\begin{aligned} P(\tau > t | X_{t_1}, \dots, X_{t_i}) &= P\left(\inf_{s \leq t_i} X_s > a \mid \inf_{t_i \leq s < t} X_s > a \mid \mathcal{F}_{t_i}\right) \Big| X_{t_1}, \dots, X_{t_i} \\ &= \Phi(t - t_i, a - X_{t_i}) P\left(\inf_{s \leq t_i} X_s > a \mid X_{t_1}, \dots, X_{t_i}\right). \end{aligned}$$

Write K_i for the second term on the right-hand-side. It can be rewritten as

$$\begin{aligned} K_i &= P\left(\inf_{s \leq t_i} X_s > a \mid X_{t_1}, \dots, X_{t_i}\right) \\ &= P\left(\inf_{s \leq t_{i-1}} X_s > a \mid X_{t_1}, \dots, X_{t_i}\right) P\left(\inf_{t_{i-1} \leq s < t_i} X_s > a \mid X_s : s \leq t_{i-1}, X_{t_i}\right) \Big| X_{t_1}, \dots, X_{t_i}. \end{aligned}$$

An analytical expression for the inside member can be obtained as follows:

$$\begin{aligned} P\left(\inf_{t_{i-1} \leq s < t_i} X_s > a \mid X_s : s \leq t_{i-1}, X_{t_i}\right) &= P\left(\inf_{t_{i-1} \leq s < t_i} X_s - X_{t_{i-1}} > a - X_{t_{i-1}} \mid X_{t_i}, X_{t_{i-1}}\right) \\ &= \exp\left(-\frac{2}{t_i - t_{i-1}}(a - X_{t_{i-1}})(a - X_{t_i})\right). \end{aligned}$$

Therefore,

$$K_i = K_{i-1} \exp\left(-\frac{2}{t_i - t_{i-1}}(a - X_{t_{i-1}})(a - X_{t_i})\right), \quad (27)$$

which can be solved recursively by using the initial condition K_1 computed in Subsection 4.1.

Hence, the general formula for the conditional default probability F_t^2 reads

$$\begin{aligned} P(\tau \leq t \mid \mathcal{F}_t) &= 1 \quad \text{if } X_{t_j} < a \text{ for at least one } t_j, t_j < t \\ &= 1 - \Phi(t - t_i, a - X_{t_i})K_i, \end{aligned}$$

where

$$\begin{aligned} K_i &= \left(1 - \exp\left(-\frac{2}{t_1}a(a - X_{t_1})\right)\right) \\ &\quad \left(1 - \exp\left(-\frac{2}{t_2 - t_1}(a - X_{t_1})(a - X_{t_2})\right)\right) \\ &\quad \dots \\ &\quad \left(1 - \exp\left(-\frac{2}{t_i - t_{i-1}}(a - X_{t_{i-1}})(a - X_{t_i})\right)\right). \end{aligned}$$

Accordingly, the \mathbf{F} -hazard process is given by

$$\begin{aligned} \Gamma_t^2 &= \infty \quad \text{if } X_{t_j} < a \text{ for at least one } t_j, t_j < t \\ &= -\ln(\Phi(t - t_i, a - X_{t_i})K_i). \end{aligned}$$

The Doob-Meyer decomposition of F^2 is

$$F_t^2 = \zeta_t^2 + (F_t^2 - \zeta_t^2),$$

where ζ^2 is an \mathbf{F} -martingale and $F^{2,c} := F^2 - \zeta^2$ is a continuous, hence predictable, increasing process. From the properties of F^2 , ζ^2 is the pure jump process defined by

$$\zeta_t^2 = \sum_{i, t_i \leq t} \Delta F_{t_i}^2.$$

Indeed, we prove below that ζ^2 is an \mathbf{F} -martingale.

Lemma 4 *The process ζ^2 is an \mathbf{F} -martingale.*

PROOF: In the case, $t_i \leq s < t \leq t_{i+1}$, it is obvious that $E(\zeta_t^2 | \mathcal{H}_s) = \zeta_s^2$ since $\zeta_t^2 = \zeta_s^2 = \zeta_{t_i}^2$, which is \mathcal{H}_s -measurable.

To end the proof, it suffices to show that $E(\zeta_t^2 | \mathcal{F}_s) = \zeta_s^2$ for $t_i \leq s < t_{i+1} \leq t < t_{i+2}$. In this case, $\zeta_s^2 = \zeta_{t_i}^2$ and $\zeta_t^2 = \zeta_{t_i}^2 + \Delta F_{t_{i+1}}^2$. Therefore,

$$\begin{aligned} E(\zeta_t^2 | \mathcal{F}_s) &= E(\zeta_{t_i}^2 + \Delta F_{t_{i+1}}^2 | \mathcal{F}_s) \\ &= \zeta_{t_i}^2 + E(\Delta F_{t_{i+1}}^2 | \mathcal{F}_s), \end{aligned}$$

and we are reduced to prove that $E(\Delta F_{t_{i+1}}^2 | \mathcal{F}_s) = 0$.

Let $s < u < t_{i+1} < v < t$. Then,

$$E(F_v^2 - F_u^2 | \mathcal{F}_s) = E(\mathbb{1}_{u < \tau \leq v} | \mathcal{F}_s).$$

When $v \rightarrow t_{i+1}, v > t_{i+1}$ and $u \rightarrow t_{i+1}, u < t_{i+1}$,

$$F_v^2 - F_u^2 \rightarrow \Delta F_{t_{i+1}}^2.$$

It follows that

$$\begin{aligned} E(\Delta F_{t_{i+1}}^2 | \mathcal{F}_s) &\rightarrow \lim_{\substack{u \rightarrow t_{i+1} \\ v \rightarrow t_{i+1}}} E(\mathbb{1}_{u < \tau \leq v} | \mathcal{F}_s) \\ &= E(\mathbb{1}_{\tau = t_{i+1}} | \mathcal{F}_s) = 0. \end{aligned}$$

△

Duffie and Lando [10] established that when the information on the market is $\mathbf{F} \vee \sigma(t \wedge \tau)$, the time of default is totally inaccessible in this filtration and admits an intensity. From the results of Elliott et al. [11], the generalized \mathbf{F} -intensity of τ is the process Λ^2 defined as

$$d\Lambda_t^2 = \frac{d(F_t^2 - \zeta_t^2)}{1 - F_{t-}^2}.$$

It can be checked that the process $(F_t^2 - \zeta_t^2, t \geq 0)$ is absolutely continuous with respect to Lebesgue measure.

5.4 Bond prices

When the information available on the market is $\mathcal{F}_t \vee \sigma(\tau \wedge t)$, and the reorganization boundary for the equityholders is also α , the price of a defaultable bond with zero recovery is given by

$$\begin{aligned} Z^2(t, T) &= e^{-r(T-t)} P(T < \tau | \mathcal{F}_t \vee \sigma(\tau \wedge t)) \\ &= \mathbb{1}_{\tau > t} e^{-r(T-t)} \frac{P(T < \tau | \mathcal{F}_t)}{P(t < \tau | \mathcal{F}_t)} = \mathbb{1}_{\tau > t} e^{-r(T-t)} \frac{\Phi(T - t_i, a - X_{t_i}) K_i}{\Phi(t - t_i, a - X_{t_i}) K_i} \\ &= \mathbb{1}_{\tau > t} e^{-r(T-t)} \frac{\Phi(T - t_i, a - X_{t_i})}{\Phi(t - t_i, a - X_{t_i})}, \quad \text{for } t_i < t < t_{i+1} < T. \end{aligned}$$

If the sequences of observation dates for the two types of information are equal, this is the same as when the information is $\mathcal{H}_t \vee \sigma(\tau \wedge t)$, that is, $Z^1(t, T) = Z^2(t, T)$. This can be explained by the Markov property of X .

With \mathcal{F}_t information only, the secondary bond market values the zero recovery bond as the discounted value of a survival without default to the maturity date:

$$\tilde{Z}^2(t, T) = \mathbb{1}_{\tau > t_i} e^{-r(T-t)} \Phi(T - t_i, a - X_{t_i}) K_i, \quad \text{for } t_i < t < t_{i+1} < T. \quad (28)$$

It is important to note, that while F_t has a single jump from zero to one at τ , F_t^1 and F_t^2 have multiple jumps at the observation dates prior to default and these jumps induce jumps in the price of the defaultable bond.

6 Analysis of credit spreads

One of the failures of the structural approach is that it produces counterfactually low spreads for the short-maturity default-risky bonds (see, e.g., Jones et al. [21]). This is a consequence of the fact that τ is a stopping time in the filtration generated by the asset prices and, in a diffusion setting, if asset value is substantially larger than the default barrier and the time to the maturity is short, the probability that it falls to the barrier is small. As a result, the short-maturity credit spreads are quite low, which is in contrast to the empirically observed spreads.

To facilitate the subsequent analysis, we examine mainly the spreads of the zero-recovery defaultable bond with the different types of information. We write $S^2(t, T)$ for the yield spread of the defaultable bond when the information is $\mathcal{F}_t \vee \sigma(\tau \wedge t)$. It can be computed by taking a log of the ratio of the defaultable to risk-free zero-coupon bond with the same maturity, that is, on the set $\{\tau > t\}$,

$$\begin{aligned} S^2(t, T) &= -\frac{1}{(T-t)} \ln \frac{Z^2(t, T)}{e^{-r(T-t)}} \\ &= -\frac{1}{(T-t)} \ln \left(\frac{\Phi(T - t_i, a - X_{t_i})}{\Phi(t - t_i, a - X_{t_i})} \right). \end{aligned}$$

If the two sequences of observation dates for the two types of information are the same, $S^2(t, T) = S^1(t, T)$ holds, where $S^1(t, T)$ is the yield spread of the defaultable bond when the information is $\mathcal{H}_t \vee \sigma(\tau \wedge t)$. However, we again emphasize that this is only true if the equityholders have the same default threshold as the bondholders.

Since $\Phi(t - t_i, z)$ is a decreasing function of $t - t_i$, between two information-arrival dates, the credit spreads are increasing in the length of the observation lag $t - t_i$. It is obvious that the spreads

are decreasing in the last observation of the log of asset value, that is, a high observed X_{t_i} lead to lower credit spreads and better credit quality of the issuer. For $X_t = X_{t_i}$, the short-term credit spreads with both types of incomplete information are higher than those in the case of complete information $S(t, T) = -\frac{1}{(T-t)} \ln(\mathbb{1}_{\tau > t} \Phi(T-t, a - X_t))$.

Figure 5 compares the short term credit spreads with full information and partial asset information and observation of survivorship for the base case parameters. We consider a low observed asset value, meaning that asset value is close to the default barrier and we have a speculative-grade issuer, who is very close to default. With partial information, the short-term credit spreads are bounded away from zero. Moreover, the term structure of credit spreads is downward sloping. For larger times to maturity, credit spreads with partial information become lower than the full information spreads.

At each observation date t_{i+1} , prior to default, the jump in the credit spread with the first type of information is a negative jump, given by

$$\Delta S^1(t_{i+1}, T) = -\frac{1}{(T-t_{i+1})} \left[\ln(\mathbb{1}_{\tau > t} \Phi(T-t_{i+1}, a - X_{t_{i+1}})) - \ln\left(\mathbb{1}_{\tau > t_{i+1}} \frac{\Phi(T-t_i, a - X_{t_i})}{\Phi(t_{i+1}-t_i, a - X_{t_i})}\right) \right].$$

It is important to note that only the last observed asset value enters into the formulas for credit spreads. This is not the case when the market has access to \mathbf{H} and \mathbf{F} only. Then, assuming the same sequence of observation dates, and using (20),

$$\tilde{S}^1(t, T) = -\frac{1}{(T-t)} \ln(\mathbb{1}_{\tau \geq t_i} \Phi(T-t_i, a - X_{t_i})).$$

If the current time is close to the last observation time, that is, $t \rightarrow t_i$, $\tilde{S}^1(t, T) \rightarrow S^1(t, T)$.

With the second type of information, using (28),

$$\tilde{S}^2(t, T) = \tilde{S}^1(t, T) - \frac{1}{(T-t)} \ln K_i.$$

Also, as $t \rightarrow t_i$, $\tilde{S}^2(t, T) \rightarrow S^2(t, T)$. However, the spread converges slower, since there is the additional term $-\frac{1}{(T-t)} \ln K_i > 0$. This shows that, with asset information only, credit spreads are increasing with decrease of the information about asset value, because $\mathbf{F} \subset \mathbf{H}$. We can conclude that, when the default time is not immediately announced, a firm with \mathbf{F} -type of information is less transparent to the the bondholders than a firm with an \mathbf{H} -type of information and has higher short-term credit spreads.

In the following figures, we plot credit spreads for the base case parameters with various last observed asset values and for different observation lags. Figure 6 plots the spreads with full information, first type of information and observation of the default event, $\mathbf{H} \vee \mathbf{D}$, and the first type

of asset information only. We consider a high observed asset value, meaning that this represents a firm with a better credit quality than in the first case. While with partial information, the short-term credit spreads are higher than in the case of full information, without information about survivorship, the spreads are much larger. Also, the term structure of credit spreads with $\mathbf{H} \vee \mathbf{D}$ information is now upward sloping.

Figure 7 displays the credit spreads with full information and information $\mathbf{H} \vee \mathbf{D}$ with different observation lags for a firm with a good credit quality, that is, a high observed last asset value ($X_{t_i} = 1.3$). In this case, the increase of the observation lag from 0.5 to 1 leads to an increase of credit spreads. Figure 8 plots the credit spreads with full information and information $\mathbf{H} \vee \mathbf{D}$ with different observation lags for a low observed last asset value ($X_{t_i} = 0.4$). In this case, the increase of the observation lag from 0.5 to 1 leads to a decrease of credit spreads. Also, the term structure of credit spreads is downward sloping. It is important to note that Figure 6 and Figure 8 present cases of very high-yield bonds. We can conclude that for speculative-grade issuers with the two types of incomplete information, the term structures of short-term credit spreads are downward sloping no matter what is the observation lag. Moreover, as Figure 8 shows, with the increases of the observation lags, the term structure of credit spreads for such firms shifts downward.

In the case with non-zero recovery, for $t_i < t < t_{i+1}$ and $t < T$, the credit spread with the first type of asset information and observation of the default time is given by

$$S^{1,R}(t, T) = -\frac{1}{(T-t)} \ln \left(\mathbb{1}_{t < \tau} \frac{\Phi(T-t_i, a-X_{t_i})}{\Phi(t-t_i, a-X_{t_i})} + \frac{\alpha \mathbb{1}_{t < \tau} e^{r(T-t_i)} e^{(\nu-\gamma)(a-X_{t_i})}}{D \Phi(t-t_i, a-X_{t_i})} \{ \Phi(\gamma, t-t_i, a-X_{t_i}) - \Phi(\gamma, T-t_i, a-X_{t_i}) \} \right) \quad (29)$$

and is obviously smaller than $S^1(t, T)$.

7 Comparison of hazard process and intensity methods

In this section, we study the properties of the compensator of the default process. Since with full asset information, τ is a \mathbf{F}^V -stopping time, $\Lambda_t = D_t = \mathbb{1}_{\tau \leq t}$ and the compensated martingale M_t vanishes. The most interesting cases are when τ is a totally inaccessible stopping time in the reference filtration and the compensated martingale M_t^i , associated with the default process, is different from zero. If we consider the first type of partial information, using the results of Elliott et al. [11], the \mathbf{H} -generalized intensity of τ is the \mathbf{H} -adapted process Λ^1 given by

$$\Lambda_t^1 = \int_{]0, t]} \frac{dF_s^{1,c}}{1 - F_{s-}^1}.$$

Figure 5: Credit spreads with full information and partial information for $X_{t_i} = X_t = 0.3$, $t = 0.5$ and observation lag $t - t_i = 0.5$

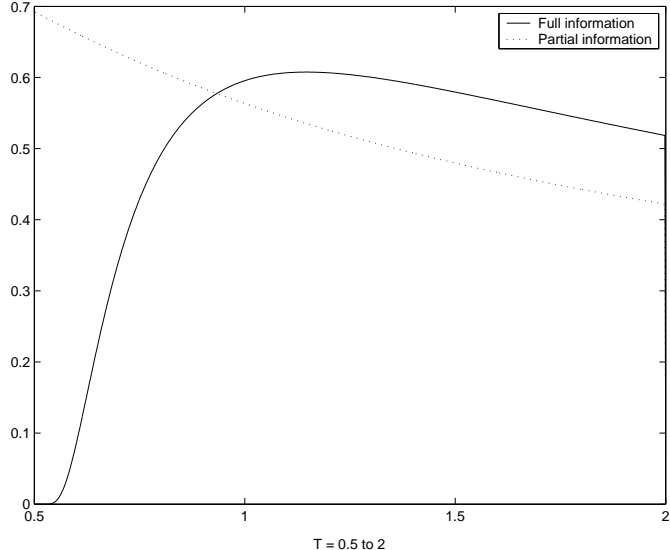


Figure 6: Credit spreads with full information and partial information for $X_{t_i} = X_t = 1.0$, $t = 0.5$ and observation lag $t - t_i = 0.5$

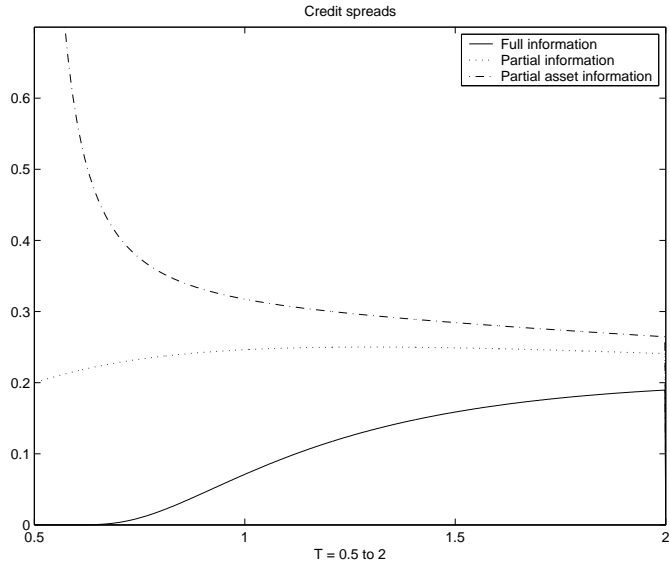


Figure 7: Credit spreads with full information and partial information for $X_{t_i} = X_t = 1.3$, $t = 1$ and observation lags $t - t_i = 0.5$ and $t - t_i = 1$, respectively

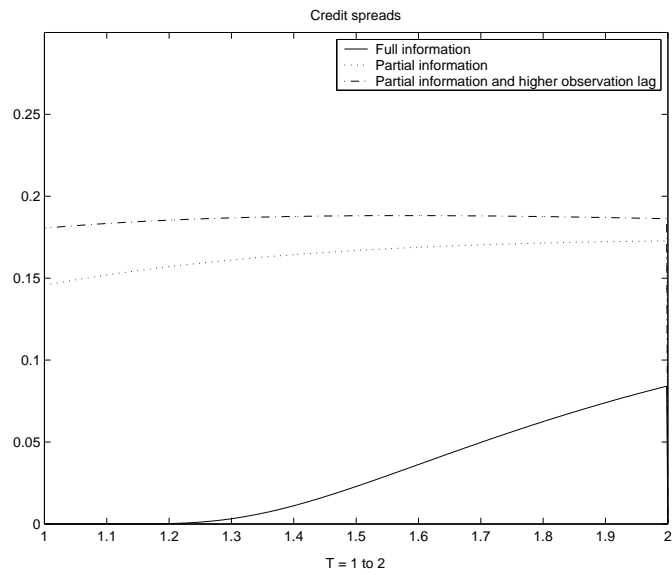
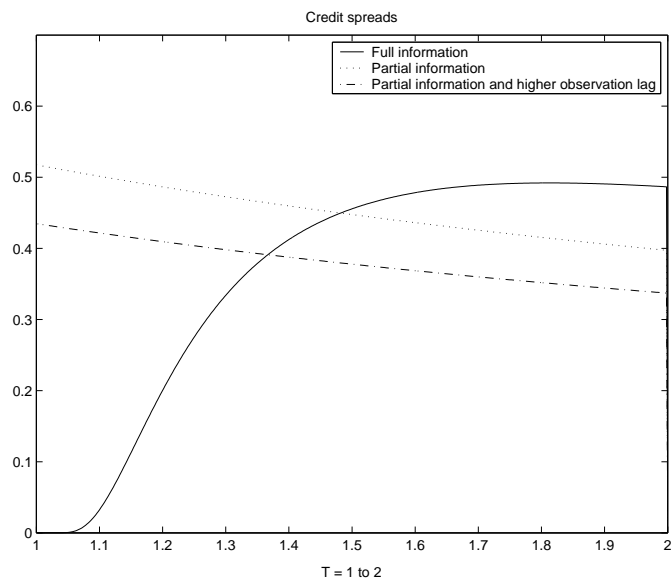


Figure 8: Credit spreads with full information and partial information for $X_{t_i} = X_t = 0.4$, $t = 1$ and observation lags $t - t_i = 0.5$ and $t - t_i = 1$, respectively



Moreover, for $t_n < t < t_{n+1}$, from the continuity of $F^{1,c}$

$$\Lambda_t^1 = \int_{]0,t]} \frac{dF_s^{1,c}}{1 - F_{s-}^1} = \sum_{i=0}^{n-1} \int_{]t_i, t_{i+1}]} \frac{dF_s^{1,c}}{1 - F_{s-}^1} + \int_{]t_n, t]} \frac{dF_s^{1,c}}{1 - F_{s-}^1}$$

and on $]t_i, t_{i+1}]$, $F^{1,c}$ is differentiable. Therefore, we can write

$$\begin{aligned} \Lambda_t^1 &= \sum_{i=0}^{n-1} \int_{]t_i, t_{i+1}]} \frac{dF_s^{1,c}}{1 - F_{s-}^{1,c} - \zeta_{t_i}^1} + \int_{]t_n, t]} \frac{dF_s^{1,c}}{1 - F_{s-}^{1,c} - \zeta_{t_n}^1} \\ &= \sum_{i=0}^{n-1} -\ln(1 - F_{s-}^{1,c} - \zeta_{t_i}^1) \Big|_{t_i}^{t_{i+1}} - \ln(1 - F_{s-}^{1,c} - \zeta_{t_n}^1) \Big|_{t_n}^t \\ &= -\sum_{i=0}^{n-1} \ln \frac{(1 - F_{t_{i+1}}^{1,c} - \zeta_{t_i}^1)}{(1 - F_{t_i}^{1,c} - \zeta_{t_i}^1)} - \ln \frac{(1 - F_t^{1,c} - \zeta_{t_n}^1)}{(1 - F_{t_n}^{1,c} - \zeta_{t_n}^1)}. \end{aligned} \quad (30)$$

The process $F_s^{1,c}$ is differentiable and $dF_s^{1,c} = f_s^{1,c} ds$. Hence, the \mathbf{H} -generalized intensity Λ^1 admits the following integral representation:

$$\Lambda_t^1 = \int_0^t \lambda_s^1 ds,$$

where the \mathbf{H} -intensity of τ is $\lambda_s^1 = \frac{f_s^{1,c}}{1 - F_{s-}^{1,c} - \zeta_{t_i}^1}$ on the interval $]t_i, t_{i+1}[$.

We have seen that the formula for the prices in the hazard process framework on the set $\{\tau > t\}$ is

$$Z^1(t, T) = e^{-r(T-t)} E \left(e^{\Gamma_t^1 - \Gamma_T^1} \Big| \mathcal{H}_t \right).$$

This is equal to

$$\begin{aligned} Z^1(t, T) &= e^{-r(T-t)} E \left(F_T^1 - F_t^1 \Big| \mathcal{H}_t \right) \\ &= e^{-r(T-t)} E \left(F_T^{1,c} - F_t^{1,c} \Big| \mathcal{H}_t \right) \end{aligned}$$

due to the martingale property of ζ^1 .

A use of the intensity process in the case when both $t = t_n$ and $T = t_N$ are observation times would lead to

$$e^{-r(T-t)} E \left(e^{\prod_{i=n}^{N-1} \frac{1 - F_{t_{i+1}}^{1,c} - \zeta_{t_i}^1}{1 - F_{t_i}^{1,c} - \zeta_{t_i}^1}} \Big| \mathcal{H}_t \right),$$

which is a different result and is much more difficult to compute.

8 Conclusion

As our results show, bond prices in the classical structural models with continuous and perfect observation of asset value are continuous. Yet, market prices of defaultable bonds display frequent

jumps at times of release of important information about the credit quality of the issuers. In our view, the two partial information models display realistic information flows to the secondary bond market. These models also yield more realistic bond prices and credit spreads. Thus, the two information-constrained models reproduce the important stylized facts that the term structures of short-term credit spreads start higher than zero and are sometimes flat or downward sloping. This is especially true for high-yield bonds. We also find that when the last observed asset value is high, that is, the firm is with a good credit quality, credit spreads are increasing with the reduction of the information sets and with the increases of the observation lags. We can conclude that firms in good financial conditions should have strong incentives to release more information to their bondholders since this would lower the costs of their debt financing. For firms with moderate financial results, release of more information will lead to lower credit spreads for bonds with short-term maturities but to higher credit spread for bonds with intermediate maturities.

As discussed in the previous sections, we have the following containment relation between the different filtrations: $\mathbf{F}^V \supset \mathbf{H} \supset \mathbf{F}$. The default time changes from a predictable stopping time in the filtration \mathbf{F}^V to a totally inaccessible time when information of the secondary bond market is reduced to \mathbf{H} or \mathbf{F} . Consequently, the compensated martingale associated with the default process is zero with full information. This is no more the case with partial information. Since in these cases the generalized intensities are absolutely continuous, τ admits risk-neutral default arrival intensities and the well developed reduced form models could be used to model the default event. However, the hazard process approach allows for a direct evaluation of conditional expectations. With the two types of information, intensities depend on asset value at the observation times. Another observation scheme (e.g. cash-flow induced default) can make them dependent on other firm-specific information.

If the information to bondholders is augmented with announcement of default and liquidation by equityholders, we have the following containment relation: $\mathbf{F}^V \supset (\mathbf{H} \vee \mathbf{D}) \supset (\mathbf{F} \vee \mathbf{D})$. Bond prices and credit spreads in the two expanded asset subfiltrations, $\mathbf{H} \vee \mathbf{D}$ and $\mathbf{F} \vee \mathbf{D}$, are the same. However, this is true only if the equityholders have the same default threshold as the bondholders and the sequences of observation dates for the two types of information are the same. In such cases, the last observed asset value is a sufficient statistics for the prices of defaultable bonds and credit spreads. The models described here could serve as a basis for a study of the two valuation approaches with more complicated asset information structures (e.g., combining discrete with continuous but noisy observations).

In addition, discontinuities associated with asset value process are also important to the extend

that, in practice, stock and defaultable bond prices have common jumps at times of release of important information about the issuer. However, because of the mathematical complexity of the first passage problem, analytical solutions for default probability and bond prices are still not available, even in the base case of full information. Finding solutions with partial information remains a challenge for future research.

It is important to note that, in practice, the information sets of investors in the secondary markets for credit-risky bonds include additional non firm-specific information. This non-firm specific information should be included in the subfiltration generated by asset prices. Such information is economy-wide information such as the short-term interest rate, industrial production, and the exchange rate. Another example of relevant non firm-specific information is the case of extraneous risk, in which the market perception of asset value dynamics changes with default of another firm within the same industry (market).

As previously pointed out, defaultable bonds are held in large portfolios by institutional investors such as investment funds. Default of a single name in their portfolio may cause them to revise their beliefs about the likelihood of default of the other issuers. For example, in the case of a single default, followed by a debt crisis, the other firms may not be able to roll-over the existing bonds and may experience financial difficulties. Then, the distributional properties of τ depend on default times of the other firms. Therefore, information about the assets and liabilities of other firms also plays a key role in determining default probability of the reference entity. It is a major challenge for future research to provide explicit computations of conditional hazard processes and valuations of defaultable bonds with such expanded information sets, consistent with the total information sets of the market. In such models, the structure of the different asset value processes and the links between them would be very important.

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