Equilibrium Asset Pricing and Portfolio Choice with Heterogeneous Preferences *

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Abstract

We study the market price of risk, the stock volatility and the hedging behavior in equilibrium of heterogeneous agents with arbitrary utility functions, consuming only at the end of the time horizon, and with the state variable following an arbitrary homogeneous diffusion process. We introduce a new notion that we call the "rate of macroeconomic fluctuations", and show that, in equilibrium, all the quantities and strategies can be characterized in terms of the dividend volatility and the interest rate volatility discounted at this rate.

We also show that both the optimal portfolio strategies and the stock price volatility can be decomposed into a myopic and a non-myopic component. The market price of risk, the myopic volatility and the myopic portfolio are determined by the present market value of future discounted volatilities of the dividend and of the interest rate. By contrast, the non-myopic volatility and non-myopic portfolio are given in terms of covariances of equilibrium quantities with the discounted dividend volatility. These representations enable us to show that, under natural cyclicality conditions, the non-myopic volatility is always positive, and the non-myopic portfolio is positive for an agent if and only if the product of his prudence and risk tolerance is less than the same product corresponding to the log agent.
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JEL Classification. D53, G11, G12

1 Introduction

What is the equilibrium prediction for the market price of risk and volatility of the market portfolio in a complete market economy populated by heterogeneous agents, and when the investment opportunity set is stochastic? And what is the hedging behavior of those agents in equilibrium? Those are the two main questions we answer in this paper.

The first main insight our results provide is that all equilibrium quantities can be characterized in terms of myopic terms and non-myopic terms. Myopic terms are determined as the *present market value* of aggregate quantities, such as dividend volatility and aggregate risk aversion. By contrast, the non-myopic terms are determined by the *future fluctuations* of aggregate quantities, or more precisely, by covariances thereof. The second insight is that, for both myopic and non-myopic terms, the aggregate quantities are priced at a discount (premium), discounted (appreciated) at a rate we call the *rate of macroeconomic fluctuations*, or RMF.

It is known (Merton 1973) that, with a CRRA representative agent with risk aversion γ consuming only at the end of the time horizon, with the dividend following a Geometric Brownian Motion (GBM) with volatility σ, and with a constant interest rate, the equilibrium market price of risk (MPR) is given by $\gamma\sigma$, the stock price volatility equals the fundamental volatility σ, and the optimal portfolio is myopic and instantaneously mean-variance efficient. However, little is known about the equilibrium dynamics when at least one of these assumptions is violated. We study equilibrium of heterogeneous agents with general utility functions, consuming only at the end of the time horizon, with stochastic interest rates, and the state variable (the aggregate dividend) following an arbitrary diffusion.[1]

[1] The simplifying assumption of no intermediate consumption is frequently used in

We first obtain a representation for equilibrium MPR. It is myopic by its nature because its current level only impacts the myopic part of the agents' portfolios. Intuitively, the size of equilibrium MPR should be determined by the size of the total future risk in the economy, that is, by the future dividend and interest rate volatilities. When the dividend volatility is stochastic, current dividend volatility, also known as fundamental volatility, may have little relation to the future volatility and hence there may be no direct relation between MPR and current volatility.

In order to specify the exact representation for MPR (and other equilibrium quantities) in our model, and its relation to the value of the total future risk, we define a new quantity, the above mentioned *rate of macroeconomic fluctuations*, RMF, that only depends on the exogenous state variable process. In a special, illustrative case, RMF is the rate at which the growth rate of the economy is changing, and in order to account for these fluctuations, equilibrium values need to be discounted (or appreciated) at that rate.

It turns out that, with the aggregate dividend encompassing a stochastic opportunity set rather than resulting from a GBM, the role of fundamental volatility is played by the market value of future dividend volatility, discounted at RMF. We call this quantity "myopic volatility" because it only depends on the present market value of dividend volatility and not on its future co-movement with other variables. Then, if the interest rate and aggregate risk aversion are constant, the following "myopic" version of continuous-time CAPM holds: the market price of risk is given by the product of risk aversion and the myopic volatility. However, if the (aggregate) risk

equilibrium asset pricing literature. See, e.g., Bick (1990), He and Leland (1993), Kogan et al. (2006). Since agents do not have to substitute consumption for investment, the interest rate is not determined in equilibrium and can be specified exogenously. This allows to isolate the interest rate effects on the equilibrium dynamics. One can contrast this to production economies, in which the market price of risk is specified exogenously and only the interest rate is determined in equilibrium. See, e.g., Dumas (1989).

aversion is constant, but the interest rate is stochastic, MPR is given by the product of the risk aversion and the myopic volatility net of the market value of the cumulative interest rate volatility, discounted at RMF. A consequence of this representation is that the procyclical interest rate decreases MPR. The reason is that, when the interest rate is procyclical, bond prices are countercyclical. Therefore, the stock, being negatively correlated with bonds, becomes more attractive as an instrument for hedging interest rate risk. This drives up the equilibrium demand for stock, and the stock price goes up, pushing returns down.

Unlike MPR, the equilibrium stock volatility is a non-myopic quantity, in general. It depends not only on the present value of future (discounted) dividend volatility, but also on the cyclical properties of MPR. More precisely, we show that the non-myopic volatility, i.e., the difference between the stock volatility and the myopic volatility, is given by the negative covariance of future discounted MPR with the aggregate dividend. The myopic/non-myopic volatility decomposition carries over to that of the optimal portfolios: the non-myopic part of the portfolio is determined by the negative covariance of the future discounted MPR with an agent-specific quantity, the difference between the agent's wealth and his risk tolerance.

The intuition behind these results is as follows. Stock volatility is determined by the sensitivity of the stock price to changes in the dividend. When MPR and the interest rate are non-stochastic, the optimal portfolios are myopic and only depend on the current value of MPR, and, therefore, the equilibrium price sensitivity is also myopic. By contrast, when MPR is stochastic, it generates non-myopic demand, determined by future fluctuations of MPR relative to the agent's future wealth and risk tolerance. This hedging demand for stocks drives the stock price up or down and thus increases or decreases stock price sensitivity and gives rise to the non-myopic component of stock volatility and of the agent's portfolio.

There are several empirical implications generated by our model that can be tested. First, our results imply that the rate of macroeconomic fluctuations is the difference between the drift of MPR and the product of MPR and its volatility.[2] We could then estimate the latter values from data and test whether the dividend is indeed not modeled well by the geometric Brownian motion. [3]

Second, as already remarked above, our results show that with the dividends which are not GBM, the standard way of comparing fundamental volatility to the volatility of the dividends is not appropriate. The natural analog of fundamental volatility in our general setting is the myopic volatility, that can be computed given the dividend volatility and RMF, as can the non-myopic volatility. We can then compare their sum to the true realized volatility, to test the prediction of our model. A similar analysis could be done for the optimal portfolios.

Third, our model results in a potentially highly nonlinear risk-return profile, as often observed empirically. This occurs because the short term stock returns are driven by the myopic volatility and not by the total volatility of the stock price, and the risk-return ratio then depends on the ratio between the total and the myopic volatility.

Another consequence of our results is that the non-myopic volatility is positive under the same conditions under which MPR is countercyclical,[4]

[2]Namely, $d\lambda_t = \lambda_t\,(\mu_t^\lambda\,dt + \sigma_t^\lambda\,dB_t)$ with the drift μ_t^λ given by $\mu_t^\lambda = \rho_t + \lambda_t\,\sigma_t^\lambda$ where ρ_t is the rate of macroeconomic fluctuations.

[3]In addition to testing whether RMF is non-zero, it would also be of interest to estimate its cyclicality properties.

[4]Namely, aggregate risk aversion is above one, and both aggregate risk aversion and macroeconomic uncertainty are countercyclical.

and can therefore be interpreted as excess volatility.[5] The intuition behind it is as follows. In good states with high expected future dividends and low MPR, the future aggregate risk aversion is relatively low, and the agent is willing to hold the stock even if the return is low. This makes the price go up high in good states and, by the same arguments, go down in bad states, and therefore drives the price volatility up. In other words, the tension between the movements in future dividend and MPR creates excess volatility, because increasing dividend increases the demand for the stock, while at the same time decreasing MPR decreases it.

This result should be contrasted with a related result of Bhamra and Uppal (2009a). They analyze equilibrium with two CRRA agents and a GBM dividend and show that the excess volatility is positive if agent's elasticity of intertemporal substitution (EIS) is not too large. Their model is different from ours because they allow for intermediate consumption and, therefore, the interest rate is determined endogenously by the agents' EIS. Our volatility decomposition result allows us to look at their result from a different perspective. Namely, there are two sources generating deviations from fundamental (myopic) volatility in their model. Countercyclical MPR drives the non-myopic volatility up, whereas the procyclical risk-free rate drives the volatility down, and the constraint of EIS being not too large diminishes the second effect.[6]

Our representation for the optimal portfolios yields precise conditions determining the sign (and size) of the agents' hedging demand. The non-

[5]Excess volatility is a well known stylized fact. See, e.g., Shiller (1981), LeRoy and Porter (1981), Mankiw, Romer, and Shapiro (1985, 1991) and West (1988).

[6]It would be of significant interest to study the validity of our general results in economies with intermediate consumption and endogenously determined interest rate. However, the equilibrium structure in such economies is much more complex and we leave it for future research. The advantage of exogenous risk-free rate is that we can isolate and study the effect of the stochastic interest rate on the dynamics of asset prices and optimal portfolios and derive general necessary and sufficient conditions for excess volatility.

myopic portfolio decouples into the MPR hedge and the interest rate (IR) hedge. We show that, when the interest rate is procyclical (countercyclical), the IR hedge is negative (positive) if and only if the agent's risk aversion is above one.[7] The nature of the MPR hedge is more complex. We show that, when MPR is countercyclical, the MPR hedge is positive if and only if the product of prudence and risk tolerance is below two. The benchmark case is the logarithmic utility for which this product is exactly equal to two, and the hedging part is, of course, zero. This effect on the hedging portfolio arises from a competition between two forces: precautionary savings, determined by the agent's prudence,[8] and risk aversion. When risk aversion is small(large) relative to prudence, the precautionary saving (risk aversion) motive dominates and hedging portfolio becomes negative (positive). Our representation of the MPR hedge allows us to isolate both effects and study their structure in detail. In particular, we show that the MPR hedge can be represented in terms of a difference between the covariance of future MPR and risk tolerance and the covariance of future MPR and wealth, namely, in terms of *wealth hedge* and *risk tolerance hedge*.

We illustrate our general results in the special case of the log-dividend following an autoregressive (Ornstein-Uhlenbeck) process, and with CRRA agents. We find explicit bounds on the market price of risk, excess volatility, risk-return tradeoff and the size of equilibrium optimal portfolios. In particular, the size of the latter three quantities depends on the difference between the highest and the lowest risk aversion. In other words, the level of the agents' heterogeneity is an important factor in the magnitude of the departure of equilibrium values in our model vs. those in the classical CCAPM.

[7] This result is known. See Detemple, Garcia and Rindisbacher (2003).

[8] See Kimball (1991).

We conclude the introduction with a discussion of related literature. Most of the work which extends standard CCAPM to heterogeneous risk preferences is done in special models with two CRRA agents only. Dumas (1989) considered a production economy of such a type. In equilibrium, the stock returns coincide with the exogenously specified returns on the production technology, and only the risk-free rate is determined endogenously. Dumas (1989) performs a detailed analysis of various properties of the economy, deriving comparative statics and dynamics results. Wang (1996) studies the term structure of interest rates in an economy populated by two CRRA agents, maximizing time-additive utility from intermediate consumption and the aggregate dividend following a GBM. Bhamra and Uppal (2009a) consider the same economy, and derive conditions under which excess volatility is positive.[9] Bhamra and Uppal (2009b) derive closed form expressions as convergent power series for an economy populated by two CRRA agents with arbitrary risk aversions, discount rates and heterogeneous beliefs. Basak and Cuoco (1998) study equilibria with two agents and limited stock market participation. Cvitanić and Malamud (2009a, 2009b) study asymptotic equilibrum dynamics with an arbitrary number of CRRA agents maximizing utility from terminal consumption, as the horizon becomes large.

Only few papers study general properties of equilibria with non-CRRA preferences and/or a general dividend process. Our paper is partially related to He and Leland (1993).[10] They also allow for an arbitrary dividend process and describe the set of viable price processes, that is, all price processes that can be attained by varying the utility of the representative agent. In

[9]The main message of Bhamra and Uppal (2009a) is that allowing the agents to trade in an additional derivative security, making the market complete, may actually increase the market volatility. Because of completeness, their equilibrium coincides with the Arrow-Debreu equilibrium of Wang (1996).

[10]See also Bick (1990) and Wang (1993).

particular, they show that, when the dividend is GBM, the market price of risk is countercyclical if and only if the risk aversion is countercyclical. Our representation allows us to prove the most general result of this kind. We show that countercyclical risk aversion is not enough in general, and countercyclicality of both dividend volatility and the rate of macroeconomic fluctuations is needed.

Berrada, Hugonnier and Rindisbacher (2007) provide necessary and sufficient conditions for zero equilibrium trading volume in a general continuous-time model with heterogeneous agents, multiple goods, and multiple securities.

Then, there are models in which heterogeneity comes from beliefs and asymmetric information, as in Basak (2005), Jouini and Napp (2009), Biais, Bossaerts and Spatt (2009) and Dumas, Kurshev and Uppal (2009). In particular, Dumas, Kurshev and Uppal also study excess volatility and non-myopic optimal portfolios in a model with two CRRA agents with identical preferences but heterogeneous beliefs. Some papers study static, one period economies with heterogeneous preferences. See, e.g., Benninga and Mayshar (2000), Gollier (2001), Hara, Huang and Kuzmics (2007). Methodologically, our paper is close to Detemple and Zapatero (1991), Detemple, Garcia and Rindisbacher (2003), Dumas, Kurshev and Uppal (2009) and Bharma and Uppal (2009a,b), as we use Malliavin calculus to derive the main results.

Our paper is also related to a recent work of Mele (2007). He studies monotonicity/concavity properties of equilibrium stock price and their relation to equilibrium volatility dynamics. He introduces a new object, the risk adjusted discount rate, and rewrites the equilibrium stock price as the present value of risk adjusted future dividends, discounted at this rate. Even though the idea of such a representation is similar in spirit to the one of this paper, there is no direct connection between the risk adjusted discount rate and the rate of macroeconomic fluctuations. The former depends on the endogenous equilibrium market price of risk, whereas the latter is an intrinsic property of the exogenously given dividend process. Furthermore, we obtain representations for the instantaneous moments of the price, and not of the price itself.

Several papers investigate, both theoretically and empirically, the non-linear relation between risk and return.[11] For example, Whitelaw (2000) shows that, with stochastic consumption opportunities and a representative CRRA agent, it is possible to generate a highly non-linear risk-return relationship. Our model provides a general expression for the risk-return profile that holds in *any* continuous-time CCAPM. Furthermore, we show that the source of this non-linearity is the relation between myopic and non-myopic volatilities.

The paper is organized as follows. In Section 2, we describe the setup and notation. In Section 3 we introduce RMF and in Section 4 we derive representations for MPR and stock volatility, and study their behavior. Section 5 is devoted to equilibrium optimal portfolios. Section 6 provides specific quantitative predictions of the general results. Section 7 illustrates the results in the special case of autoregressive log-dividend process and CRRA agents. Section 8 concludes.

[11] See Glosten, Jagannathan and Runkle (1993) and Harvey (2001).

2 Setup and Notation

2.1 The Model

We consider a standard setting similar to that of Wang (1996). The economy has a finite horizon and evolves in continuous time. Uncertainty is described by a one-dimensional, standard Brownian motion $B_t,\ t \in [0\,,\,T]$ on a complete probability space $(\Omega, \mathcal{F}_T, P)$, where $\mathcal{F}$ is the augmented filtration generated by B_t. There is a single share of a risky asset in the economy, the stock, which pays a terminal dividend D_T such that

$$D_t^{-1}\, dD_t \;=\; \mu^D(D_t)\, dt \;+\; \sigma^D(D_t)\, dB_t.$$

This diffusion process lives on $(0, +\infty)$ and $\sigma^D(D_t) > 0$.[12]

We also assume that a zero coupon bond with instantaneous risk-free rate $r_t \;=\; r(D_t)$ is available in zero net supply.[13] In particular, Ito's formula implies that

$$dr(D_t) \;=\; \mu^r(D_t)dt \;+\; \sigma^r(D_t)dW_t$$

with

$$\sigma^r(D_t) \;=\; r'(D_t)\, D_t\, \sigma^D(D_t)\,.$$

Note that $|\sigma^r|$ is the instantaneous volatility of the risk-free rate and the sign of (σ^r) coincides with the instantaneous correlation of the interest rate with the dividend D_t. We say that the interest rate is procyclical if σ^r is positive, i.e., $r'(D_t) > 0$, and countercyclical otherwise.

There are K competitive agents behaving rationally, and agent k is initially endowed with $\psi_k \;>\; 0$ shares of stock, an the total supply of the stock

[12]We assume that μ^D and σ^D are such that a unique strong solution exists, and also that $\mu^D \in C^1(\mathbb{R}_+)$, $\sigma^D \in C^2(\mathbb{R}_+)$. In general, whenever we use a derivative of a function, we implicitly assume it exists.

[13]We assume that the function $r(x)$ is C^2 and bounded from below.

is normalized to one,

$$\sum_{k=1}^{K} \psi_k \;=\; 1.$$

Agent k chooses portfolio strategy $\pi_{k\,t}$, the portfolio weight at time t in the risky asset, as to maximize expected utility[14]

$$E\left[u_k(W_{k\,T})\right]$$

of its final wealth W_{kT}, where the wealth W_{kt} of agent k evolves as

$$dW_{kt} \;=\; W_{kt}\left[r_t\,dt \;+\; \pi_{k\,t}\,(S_t^{-1}\,dS_t \;-\; r_t\,dt)\right].$$

In this equation, S_t is the stock price at time t. The instantaneous drift and volatility of the stock price S_t are denoted by μ_t^S and σ_t^S respectively, so that

$$S_t^{-1}\,dS_t \;=\; \mu_t^S dt \;+\; \sigma_t^S dB_t.$$

The market price of risk (MPR) λ_t is given by

$$\lambda_t \;=\; \frac{\mu_t^S \;-\; r_t}{\sigma_t^S}\,.$$

2.2 Equilibrium

Definition 2.1. *We say that the market is in equilibrium if the agents behave optimally and both the risky asset market and the risk-free market clear.*

It is well known that the above financial market is complete, if the volatility process σ_t^S of the stock price is almost everywhere strictly positive.[15]

When the market is complete, there exists a unique state price density

[14] We assume that u_k is $C^3(\mathbb{R}_+)$ and satisfies standard Inada conditions.

[15] This can be verified under some technical regularity conditions on the model primitives. See, Hugonnier, Malamud and Trubowitz (2009).

process $\xi = (\xi_t)$ such that the stock price is given by

$$S_t = \frac{E_t[\xi_T D_T]}{\xi_t}. \quad (1)$$

where

$$\xi_t = e^{-\int_0^t r_s\, ds} M_t$$

and M_t is the density process of the equivalent martingale measure Q,

$$\left(\frac{dQ}{dP}\right)_t = M_t = E_t[\, M_T\,]\,.$$

Thus, we can rewrite (1) in the form

$$S_t = E_t^Q[\, e^{-\int_t^T r_s\, ds} D_T\,]\,. \quad (2)$$

Because of the market completeness, any equilibrium allocation is Pareto-efficient and can be characterized as an Arrow-Debreu equilibrium. See, e.g. Duffie and Huang (1986), Wang (1996).[16]

Introduce the inverse of the marginal utility

$$I_k(x) := (u_k')^{-1}(x). \quad (3)$$

It is well known that in this complete market setting the optimal terminal wealth is of the form

$$W_{k\,T} = I_k(y_k\, \xi_T)$$

[16]Because the endowments are co-linear (all agents hold shares of the same single stock), it can be shown that, under some conditions on the agents' utility functions, the equilibrium is in fact unique up to a multiplicative factor, and unique if we fix the risk-free rate. See, e.g., Dana (1995), Dana (2001). If the dividend is neither bounded away from zero nor from infinity, some additional care is needed to verify the existence of equilibrium. See, e.g., Dana (2001) and Malamud (2008). We implicitly assume throughout the paper that an equilibrium exists.

where y_k is determined via the *budget constraint*[17]

$$E[I_k(y_k\,\xi_T)\,\xi_T] \;=\; W_{k\,0} \;=\; \psi_k\,S_0 \;=\; \psi_k\,E[\xi_T\,D_T].$$

Since in equilibrium the final wealth amounts of all the agents have to sum up to the aggregate dividend, equilibrium SDF ξ_T needs to solve the equation

$$\sum_{k=1}^{K} \; I_k(y_k\,\xi_T) = \; D_T\,. \tag{4}$$

3 Rate of Macroeconomic Fluctuations

In this section we introduce a new notion, the rate of macroeconomic fluctuations. This rate will play a crucial role in the sequel and will appear in expressions for all equilibrium quantities (market price of risk, volatility and optimal portfolios).

There are two forces driving the stochastic dynamics of the dividend D_t : stochastic volatility $\sigma^D(D_t)$ and stochastic drift $\mu^D(D_t)$. The volatility σ^D determines the *size of macroeconomic fluctuations.* In order to understand the role of μ^D, we will "extract" all the stochastic volatility from D_t.

Fix an $x_0 \in \mathbb{R}_+$ and introduce the function

$$F(x) \;\stackrel{def}{=}\; \int_{x_0}^{x} \frac{1}{y\,\sigma^D(y)}\,dy\;,\;\; x\;>\;0.$$

Then, a direct calculation shows that the diffusion coefficient of the process

$$A_t \;=\; F(D_t)$$

is equal to one. Since A_t is also an Ito diffusion, its dynamics are given in

[17]We assume a unique such y_k exists.

the form

$$dA_t \;=\; C(A_t)\,dt \;+\; dB_t\,.$$

Definition 3.1. *The negative instantaneous expected change of the volatility-extracted process A_t will be referred to as the rate of macroeconomic fluctuations, or RMF, and denoted ρ_t. More precisely, we define*

$$\rho_t \;=\; -\,C'(A_t)\,.$$

Note that RMF is a function of the current dividend: if we define the function

$$c(x) \;=\; c^D(x) \;\stackrel{def}{=}\; -C'(F(x))$$

then

$$\rho_t \;=\; c^D(D_t)$$

It follows directly from the definition that the function c is invariant under functional transformations. [18] That is, if $D_t \;=\; g(\tilde{D}_t)$ for some process $\tilde{D}_t$ and a smooth, increasing function g then

$$c^D(g(x)) \;=\; c^{\tilde{D}}(x)\,.$$

In particular, the following is true

Corollary 3.1. *RMF ρ_t is constant if and only if the dividend D_t is a a smooth and strictly increasing function of an autoregressive process*

$$dA_t \;=\; (a\,-\,b\,A_t)\,dt \;+\; dB_t. \tag{6}$$

[18]It is possible to verify that function $c(x)$ is given by

$$c(x) \;=\; -x\,(\mu^D)'(x) \,+\, x\,(\sigma^D)'(x)\,\sigma^D(x)^{-1}\,\mu^D(x) \\ +\,(\sigma^D)'(x)\,\sigma^D(x)\,x \,+\, 0.5\,(\sigma^D)''(x)\,x^2\,\sigma^D(x). \tag{5}$$

See Appendix. We note that the right-hand side of (33) appears also in the paper Detemple, Garcia and Rindisbacher (2003) in a partial equilibrium setting, as an auxiliary process that helps improve the computational efficiency, without having a direct economic interpretation.

In this case, $\rho_t \; = \; b$.

As our benchmark example throughout the paper, we will consider a simple generalization of the Geometric Brownian Motion (GBM) as the dividend process, by having $A_t = \log D_t$ satisfy (6). Then, if $b > 0$, the growth rate of the economy (the log dividend) is mean reverting. On the other hand, if $b < 0$, then the drift of the growth rate of the economy has a term increasing in absolute value of the growth rate, at the rate b.

4 Non-Myopic Volatility and Risk-Return Tradeoff

In our economy the markets are complete. This implies that the prices coincide with those in an artificial economy populated by a single, representative agent with a utility function U (see Duffie 2001). The equilibrium stochastic discount factor equals the marginal utility of the representative agent, evaluated at the aggregate endowment,

$$\xi_T \; = \; U'(D_T). \tag{7}$$

That is, the function $U'(x)$ satisfies the equation

$$\sum_k I_k(y_k \, U'(x)) \; = \; x. \tag{8}$$

Let

$$\gamma^U(x) \; = \; -\frac{x \, U''(x)}{U'(x)}$$

be the relative risk aversion of the representative agent.

Definition 4.1. *We will refer to*

$$\sigma^D(t\,,\,\tau) \; \stackrel{def}{=} \; e^{-\int_t^\tau \rho_s \, ds} \, \sigma^D(D_\tau)$$

as the discounted dividend volatility, to

$$\sigma^r(t\,,\,\tau)\ \stackrel{def}{=}\ e^{-\int_t^\tau\,\rho_s\,ds}\,\sigma^r(D_\tau)$$

as the discounted interest rate volatility, to

$$\lambda(t\,,\,\tau)\ \stackrel{def}{=}\ e^{-\int_t^\tau\,\rho_s\,ds}\,\lambda_\tau$$

as the discounted market price of risk, and to

$$\sigma_t^{\text{myopic}}\ \stackrel{def}{=}\ E_t^Q\left[\,\sigma^D(t\,,\,T)\,\right] \tag{9}$$

as the (equilibrium) myopic volatility.

We call *myopic* all the quantities that are determined by the present market value of their (possibly discounted) end-of-horizon value. In other words, their current value does not depend on their future co-movement with other market variables, but only on the "myopic estimate" of the future value.

4.1 Equilibrium MPR

The next result provides a representation of the market price of risk in terms of the quantities, introduced in Definition 4.1 and highlights the role of myopic volatility.

Theorem 4.1. *The equilibrium market price of risk is given by*

$$\lambda_t\ =\ E_t^Q\,[\,\lambda(t\,,\,T)\,]\ -\ E_t^Q\left[\int_t^T\ \sigma^r(t,s)\,ds\right] \tag{10}$$

where

$$\lambda(t\,,\,T)\ = e^{-\int_t^T\,\rho_s\,ds}\,\lambda_T\ =\ e^{-\int_t^T\,\rho_s ds}\,\gamma^U(D_T)\,\sigma^D(D_T)\,. \tag{11}$$

In particular, if the representative agent's risk aversion $\gamma^U=\gamma$ is constant, *then*

$$\lambda_t \;=\; \gamma\,\sigma_t^{\rm myopic} \;-\; E_t^Q\left[\int_t^T \sigma^r(t,s)\,ds\right]. \tag{12}$$

It is instructive to compare the result of Theorem 4.1 with the analogous result for Merton's CCAPM. When the interest rate is constant, the dividend follows a GBM with volatility σ and $\gamma^U = \gamma$, we have

$$\lambda_t \;=\; \gamma\,\sigma.$$

Formula (12) shows that, when the dividend is not a GBM, the role of the fundamental volatility σ is played by the myopic volatility $\sigma_t^{\rm myopic}$. The latter is given by the present market value of future dividend volatility, discounted at the rate ρ_t, and thus may depend on the exact dynamic properties of D_t in a nontrivial way. In particular, dividend volatility by itself is not enough for determining the size of equilibrium MPR, we also need to know the rate ρ_t. For example, if ρ_t is always positive(negative), the level of fundamental volatility leads to overestimating (underestimating) the size of MPR.

In order to understand the intuition behind this discounting, consider a simple modification of Merton's CCAPM, for which the log dividend follows (6) with $b > 0$ and both $\gamma^U = \gamma$ and r are constant. Then, $\rho_t = b$ and

$$\lambda_t \;=\; \sigma\,e^{-b\,(T-t)}\,.$$

By Theorem 4.1, both the equilibrium riskiness of the stock and the MPR are determined by the myopic volatility. On the other hand, the stock riskiness is determined by its sensitivity to changes in the state variable D_t. If the stock price moves a lot in response to a change in the dividend, stock is risky. If the sensitivity is small, stock price fluctuations are small and the stock is not risky. By (2), the stock price is the expectation (under the risk neutral measure) of future dividends. If $b > 0$, log dividends are mean reverting, and therefore, the expectation of future dividends will change less than in the case $b = 0$. Consequently, stock price sensitivity to dividend changes is smaller, stock riskiness is smaller and equilibrium MPR is smaller. However, as the time t approaches terminal horizon T, there is not enough time left for the dividends to mean-revert and the sensitivity of expected future dividends to changes in current dividend grows exponentially at the rate b, equal to the rate of mean-reversion. Thus, stock riskiness also grows exponentially fast, and converges to the riskiness of the dividend when $t \uparrow T$.

The situation is opposite when the rate $\rho_t = b$ is negative. Then, the log-dividend is non-stationary and every small fluctuation around the mean may force the dividend to deviate substantially. Hence, stock is riskier than in the case $b = 0$, and MPR goes up. However, as time t approaches T, there is not enough time for the dividend to deviate, riskiness decreases at the rate b and equilibrium MPR goes down.

When the interest rate is stochastic, another term, $E_t^Q \left[\int_t^T \sigma^r(t,s)\, ds \right]$, appears in (12). To understand the intuition behind this term, suppose that the interest rate is procyclical. Then, the bond prices are countercyclical. Thus, bonds are expensive when stocks are cheap. This drives the demand for stocks up, pushing the equilibrium stock price up, and, consequently, the equilibrium stock returns and MPR decrease. Note that there is empirical evidence that interest rates are procyclical. However, there is also evidence that interest rates are negatively correlated with some market indices (such as, e.g., S&P 500). Thus, for example, if we interpret the market portfolio as the S&P 500, we may assume that the interest rate is countercyclical, which will lead to an increase in the equilibrium MPR. Note however that the empirical interest rate volatility is very small and therefore the correction to MPR, arising from stochastic interest rates, is also very small.

The following corollary is a direct consequence of Theorem 4.1.

Corollary 4.1. *Assume that r is constant.*[19] *Under the equilibrium risk neutral measure, the drift of MPR is independent of the representative agent's utility, and is always equal to ρ_t.*

Corollary 4.1 has direct empirical implications, because it is *model-independent.* More precisely, it implies that we have the dynamics

$$d\lambda_t \;=\; \lambda_t\,(\mu_t^\lambda\,dt \;+\; \sigma_t^\lambda\,dB_t)$$

in which the drift μ_t^λ is given by

$$\mu_t^\lambda \;=\; \rho_t \,+\, \lambda_t\,\sigma_t^\lambda.$$

If we estimate MPR λ_t, its volatility σ_t^λ and its drift μ_t^λ from data, we immediately obtain an estimate for the rate ρ_t, because

$$\rho_t \;=\; \lambda_t\,\sigma_t^\lambda \;-\; \mu_t^\lambda$$

In particular, if the estimate were significantly different from zero, this would provide evidence that the dividend is not GBM.

4.2 Equilibrium volatility

By contrast to the MPR, stock price volatility is a non-myopic object and depends on future joint co-movement of macro variables with the aggregate dividend. The following is true:

[19] An analogous result also holds when r is stochastic, but it involves an additional term.

Theorem 4.2. *The equilibrium stock price volatility is given by*

$$\sigma_t^S \;=\; \sigma_t^{\text{myopic}} \;+\; \sigma_t^{\text{nonmyopic}} \;-\; E_t^Q\left[\int_t^T \sigma^r(t\,,\,s)\,ds\right]\,,$$

where[20]

$$\sigma_t^{\text{nonmyopic}} \;=\; -\,\frac{1}{S_t}\,\text{Cov}_t^Q\left(\big(\lambda(t\,,\,T)\;-\;\sigma^D(t\,,\,T)\big)\,,\;e^{-\int_t^T r(D_s)ds}\,D_T\right)\,. \quad (13)$$

Furthermore, σ_t^S is positive if r' is sufficiently small (the interest rate is not highly procyclical).

Consider first our benchmark extension of Merton's CCAPM when r, γ^U are constant, but the log dividend follows (6). Then, σ^D and the rate $\rho_t = b$ are constant and the non-myopic volatility $\sigma_t^{\text{nonmyopic}}$ vanishes. Consequently, by Theorems 4.1 and 4.2, the risk-return tradeoff coincides with that of Merton's CCAPM:

$$\lambda_t \;=\; \gamma\,\sigma_t^S.$$

However, the stock volatility does not coincide with the fundamental dividend volatility. Namely,

$$\sigma^S \;=\; \sigma_t^{\text{myopic}} \;=\; e^{-b(T-t)}\sigma^D\,.$$

As we discussed above, stock volatility will be different than the dividend volatility - higher if the log dividend is non-stationary (i.e., $b < 0$) and lower if the log dividend is stationary and mean-reverting (i.e., $b > 0$). The reason is that stock price volatility is nothing but the sensitivity of expected future dividends to changes in D_t (see, (14)). This sensitivity is large (small) if the log dividend is non-stationary (stationary). See discussion after Theorem 4.1.

Expression (13) implies that, if the interest rate is constant, the spread

[20]We use $\text{Cov}_t^Q(X,Y)$ to denote conditional covariance of random variables X and Y under the equilibrium risk-neutral measure Q.

between the stock price volatility and the myopic volatility is given by the covariance of the future discounted market price of risk net of the discounted future dividend volatility with the discounted aggregate dividend. The term "nonmyopic" is most easily justified if we recall that

$$\lambda(t\,,\,T)\;-\;\sigma^D(t\,,\,T)\;=\;\big(\gamma^U(D_T)\;-\;1\big)\,\sigma^D(t\,,\,T)\,.$$

Therefore, generically, the nonmyopic volatility vanishes if and only if $\gamma^U = 1$, that is, precisely when the representative agent is completely myopic. Hence, the nature of the nonmyopic volatility is necessarily driven by the agents' nonmyopic equilibrium behavior. When MPR is stochastic, nonmyopic agents increase or decrease stock investment depending on the expected future fluctuations of MPR. The equilibrium hedging demand raises or decreases the total equilibrium demand for stocks and therefore drives the equilibrium stock price up or down. Since D_t is the single state variable in our model, standard results imply that $S = S(t, D_t)$ is a smooth function[21] of D_t and therefore, by Ito's formula, we get

$$\sigma_t^S\;=\;\sigma^D(D_t)\,D_t\,\frac{\partial}{\partial D_t}\,\log S(t, D_t) \tag{14}$$

Thus, stock price volatility is nothing but the sensitivity of the stock price to the changes in the dividend. Since equilibrium optimal portfolios respond to changes in D_t in a nonmyopic way, so does the equilibrium stock price, giving rise to nonmyopic volatility.

Theorems 4.1 and 4.2 have direct implications for the dynamics of risk-return tradeoff. In Merton's CCAPM, MPR and the volatility of the market portfolio are linearly related, with the slope of risk-return tradeoff being equal to the aggregate risk aversion. However, empirical evidence suggests

[21] Under some technical conditions.

that expected stock returns are weakly related to volatility. The results of Glosten, Jagannathan and Runkle (1993) (see also Whitelaw (2000) and Harvey (2001)) suggest that the relation between expected returns and volatility is highly complex and non-linear. Theorems 4.1 and 4.2 imply that the nonlinear risk-return relation is driven by the nonlinear relation between the myopic and the non-myopic volatility. Indeed, consider the benchmark case when the representative agent's risk aversion is constant, as is the interest rate r. Then, the risk-return ratio is given by

$$\frac{\lambda_t}{\sigma_t^S} \;=\; \gamma\,\frac{\sigma_t^{\text{myopic}}}{\sigma_t^{\text{myopic}}\,+\,\sigma_t^{\text{nonmyopic}}}\,. \tag{15}$$

Thus, calculating the risk-return ratio amounts to calculating the ratio of the nonmyopic and the myopic volatility. One can explore empirically the ratio of myopic and nonmyopic volatilities using representations (9) and (13). It is particularly convenient that the latter do not depend on the utility of the representative agent and can be directly estimated from the data using the empirical pricing kernel (see, Rosenberg and Engle (2002)).

5 Equilibrium Optimal Portfolios

In this section we will derive various properties of equilibrium optimal portfolios. Let

$$U_{k\,t}(x) \;=\; \sup_{\pi}\, E_t\left[\,u_k(W_{kT})\,|\,W_{kt} \;=\; x\,\right]$$

be the value function of agent k[22] and introduce

$$\gamma_{k\,t} = \gamma_{k\,t}(W_{kt}) \;\stackrel{def}{=}\; -\,\frac{W_{kt}\,U''_{k\,t}(W_{kt})}{U'_{k\,t}(W_{kt})}$$

[22]Note that $U_{k\,t}$ depends on D_t but we suppress this dependence.

the *effective relative risk aversion* of agent k at time t. It is known (see Merton 1971), that, when the investment opportunity set is non-stochastic, the optimal portfolio is myopic, instantaneously mean-variance efficient and is given by

$$\pi_{k\,t}^{\text{myopic}} \stackrel{def}{=} \frac{\lambda_t}{\gamma_{k\,t}\,\sigma_t^S}\,.$$

In our model, we have

Proposition 5.1. *We have*

$$\gamma_{k\,t} = \frac{W_{k\,t}}{E_t^Q[\gamma_{kT}^{-1}\,e^{-\int_t^T r(D_s)ds}\,W_{k\,T}]}. \tag{16}$$

Note that representation (16) shows that the effective absolute risk tolerance

$$R_{kt} = \frac{W_{k\,t}}{\gamma_{k\,t}}$$

(i.e., the risk tolerance of the value function) is a martingale under the risk-neutral measure. Then, the above expression for the myopic part of the portfolio can be written as

$$W_{k\,t}\pi_{k\,t}^{\text{myopic}} = \frac{\lambda_t}{\sigma_t^S}R_{kt} = \frac{\lambda_t}{\sigma_t^S}E_t^Q[R_{kT}]\,.$$

The last equation shows that the absolute amount that the myopic portfolio component holds in the risky asset in equilibrium is equal to the current return-risk ratio times the market value of the future absolute risk tolerance, thus justifying the name "myopic".

We will denote

$$\pi_{k\,t}^{\text{hedging}} = \pi_{k\,t} - \pi_{k\,t}^{\text{myopic}}$$

and refer to it as the hedging portfolio. This is the non-myopic component of the optimal portfolio that the agent uses to hedge against (or, take advantage of) the future fluctuations in the stochastic opportunity set. The main result of this section is the following

Theorem 5.1. *We have*

$$\pi_{k\,t}^{\text{hedging}} \;=\; \pi_{k\,t}^{\text{MPR hedge}} \;+\; \pi_{k\,t}^{\text{IR hedge}}$$

with the market price of risk (MPR) hedging component

$$\begin{aligned}\pi_{k\,t}^{\text{MPR hedge}} \;&=\; -\,\frac{1}{\sigma_t^S\,W_{k\,t}}\times\\ &\text{Cov}_t^Q\left(\lambda(t\,,\,T)\,,\,e^{-\int_t^T r(D_s)ds}\,(W_{k\,T}\;-\;R_{k\,T})\right).\end{aligned} \tag{17}$$

and the interest rate (IR) hedging component

$$\pi_{k\,t}^{\text{IR hedge}} \;=\; \frac{1}{\sigma_t^S}\left(\frac{1}{\gamma_{k\,t}}\;-\;1\right)\,E_t^Q\left[\int_t^T\;\sigma^r(t,\tau)\,d\tau\right]\,. \tag{18}$$

Formulae (17) and (18) provide closed-form expressions for equilibrium hedging portfolios and show that the MPR hedge and the IR hedge have a very different nature. The IR hedge is a product of three simple terms: one that depends on the current level of agent's effective risk aversion, another given by the market value of the (discounted) cumulative interest rate volatility, and a third term, the inverse of the total stock volatility. In particular, if the interest rate is procyclical,[23] IR hedge is positive (negative) when effective risk aversion is below (above) one. Furthermore, IR hedges of two CRRA agents differ only by a constant multiple.[24]

The MPR hedge is more complex. It is determined by joint fluctuations of the discounted future MPR with agent-specific characteristics: his future wealth net of his absolute risk tolerance. Suppose for simplicity that the

[23]but not too procyclical so that σ^S is positive.

[24]By (16), $\gamma_{kt}\;=\;\gamma_k$ if agent k has a CRRA utility. This also follows from a simple homogeneity argument.

interest rate is constant. We can then write (17) as

$$e^{r(T-t)}\,\sigma_t^S\,W_{k\,t}\,\pi_{k\,t}^{\text{MPR hedge}} \;=\; -\text{Cov}_t^Q(\lambda(t\,,\,T)\,,\,W_{k\,T}) + \text{Cov}_t^Q(\lambda(t\,,\,T)\,,\,R_{k\,T})\,. \tag{19}$$

Both terms on the right-hand side have a natural interpretation. The first one is the covariance of agent's wealth with the (discounted) MPR. We will refer to it as the wealth hedge. Similarly, the second one will be referred to as the risk tolerance hedge.

To understand the nature of the wealth hedge, suppose that future MPR and the agent's wealth are positively correlated. When the wealth is low, any additional unit of wealth is valuable to the agent. Since MPR is also low in those states, the agent prefers to invest in the risk-free asset, because it offers a better hedge against the low-wealth states. This generates a negative hedging demand. When the agent's wealth is high, the utility of getting an additional unit of wealth is low and therefore, even though MPR is high in those states, it does not make stocks that much more attractive. [25]

To illustrate the intuition behind the risk tolerance hedge, suppose that MPR and risk tolerance are positively correlated. Then, when risk tolerance is high, the agent is willing to take more risk and this effect is magnified since MPR is high in those states, driving the demand for stock up. When risk tolerance is low, MPR is also low. This generates an incentive to hedge against those states and reduce the long position in the stock, but not go short too much, because of low risk tolerance. As a consequence, stock investment is altogether more attractive and this generates a positive hedging demand.[26] If we adopt Arrow's hypothesis that absolute risk aversion is monotone decreasing in wealth, risk tolerance $R_{k\,T} = R_k(W_{k\,T})$ is increasing

[25]The argument for the case of negative correlation between wealth and MPR is analogous.

[26]The case when MPR and risk tolerance are negatively correlated is analogous.

in wealth and therefore, generally speaking, the wealth hedge and the risk tolerance hedge will have opposite signs and represent competing effects for determining the size of the hedging demand. Note also that, if the agent's utility is of CARA (exponential) class, R_{kT} is constant and, consequently, the risk tolerance hedge vanishes.

There is a clear similarity between expressions (17) and (13). in fact, the hedging demand (17) is the source of nonmyopic volatility (13). Both of them arise as a nonmyopic response to the future fluctuations of MPR. Similarly to (13), expression (17) does not directly depend on the utility function of the representative agent. Consequently, one could also use the empirical pricing kernel method of Rosenberg and Engle (2002) to calculate the size of the optimal hedging portfolio.

6 Quantitative Implications

In this section we derive several specific quantitative implications of our general results above. We start with a simple bound on the equilibrium market price of risk. Let

$$\gamma_k^{\inf}\ ,\ \gamma_k^{\sup}$$

denote the infinum and supremum of the relative risk aversion of agent k. Then,[27]

$$\min_k \gamma_k^{\inf} \ \leq\ \gamma^U(x)\ \leq\ \max_k \gamma_k^{\sup} \qquad \text{for all } x\,.$$

Therefore, Theorem 4.1 immediately yields

Proposition 6.1. *Suppose that the risk free rate r is constant. Then, the*

[27]It is known (see, Wilson (1968) and Hara, Huang and Kuzmics (2007)) that the representative agent's risk aversion is a weighted average of individual risk aversions.

equilibrium market price of risk λ_t satisfies

$$\min_k \gamma_k^{\inf} \leq \frac{\lambda_t}{\sigma_t^{\text{myopic}}} \leq \max_k \gamma_k^{\sup}.$$

When the relation between risk and return is linear, the line $\lambda = \gamma\sigma$ in the (λ, σ)-plane is often referred to as the security market line. Proposition 6.1 shows that, when the risk aversion of the representative agent is stochastic, there are two security market lines in the $(\lambda, \sigma^{\text{myopic}})$ plane and the equilibrium risk-return profile stays inside the stripe between them. This phenomenon is similar to that discovered by Stulz (1981) in the context of an equilibrium model of segmented markets and can be tested empirically, providing empirical lower and upper bounds for risk aversion in the economy.

To address further dynamic properties of equilibrium, we will need to make additional assumptions about the evolution of state variables. It has become common in the literature to use consumption volatility as a measure of macroeconomic uncertainty. There is empirical evidence that macroeconomic uncertainty is countercyclical. See, e.g., French and Sichel (1993), Kim et al. (2009).

Definition 6.1. *We say that our model exhibits countercyclical macroeconomic uncertainty if the dividend volatility σ_t^D is countercyclical and the rate of macroeconomic fluctuations ρ_t is procyclical.*[28]

This definition is natural because volatility σ^D and $-\rho_t$ determine respectively the size and the speed of macroeconomic fluctuations. From now on we make the following

Assumption 6.1. *The economy exhibits countercyclical macroeconomic uncertainty and the aggregate risk aversion γ^U is countercyclical.*

[28]We call an economic variable countercyclical if it is monotone decreasing in D_t and procyclical otherwise.

By Definition 3.1, the economy exhibits countercyclical macroeconomic uncertainty if and only if $\log D_t = f(A_t)$ where f is increasing and concave, and the process A_t has constant volatility and concave drift. For example, a simple family of such process arises when

$$dA_t = (a - b\,A_t)\,dt + \sigma\,dB_t.$$

and the dividend process is given by $D_t = e^{f(A_t)}$ for some increasing and concave function f.

The assumption of countercyclical risk aversion is not unnatural, as there is strong empirical evidence supporting it; see, e.g., Campbell and Cochrane (1999), Smith and Whitelaw (2009). Furthermore, countercyclical aggregate risk aversion can be generated from simple microeconomic assumptions about individual agents preferences. In fact, it is known that, if all agents in the economy have (heterogeneous) CRRA preferences, the representative agent's utility will have a decreasing (i.e., countercyclical) relative risk aversion (DRRA). See, Benninga and Mayshar (2000), Cvitanić and Malamud (2009b). A modification of the arguments from the latter paper implies also that the following slight generalization is true:

Proposition 6.2. *If all agents have DRRA preferences then the aggregate risk aversion is countercyclical.*

In fact, it is possible to show that Proposition 6.2 still holds if individual risk aversions are sufficiently heterogeneous and "not too increasing".

It is a conventional wisdom that countercyclical risk aversion generates countercyclical market price of risk.[29] The following result confirms this intuition.

[29]See Campbell and Cochran (1999). There is also empirical evidence supporting countercyclicality of both market price of risk and aggregate risk averion. See Fama and French (1989), Ferson and Harvey (1991).

Proposition 6.3. *Under Assumption 6.1, the market price of risk is countercyclical if both r' and r'' are sufficiently small.*[30]

He and Leland (1993) show that, when r is constant and the dividend follows a geometric Brownian motion, MPR is countercyclical if the aggregate risk aversion is countercyclical. Proposition 6.3 provides general sufficient conditions for MPR countercyclicality. These conditions are tight, in the sense that even if dividend volatility is constant and risk aversion is countercyclical, a sufficiently countercyclical rate of macroeconomic fluctuations can generate procyclical behavior of MPR.

The requirement that r' and r'' be small is not too restrictive, given that, empirically, interest rate volatility is very small, especially relative to stock volatility. However, from a theoretical perspective, it is interesting to note that the effect of stochastic interest rates on MPR might lead to unexpected equilibrium dynamics if countercyclicality of the aggregate risk aversion is not strong enough. Consider for example the case when γ^U is constant γ and the dividend is GBM (as in Merton (1973)). Then, MPR is given by

$$\lambda_t \;=\; \sigma\left(\gamma \,-\; E\left[\int_t^T r'(D_\tau)\, D_\tau\, d\tau\right]\right)$$

Consequently, the dynamics of MPR are determined exclusively by the interest rate process. In particular, if $r'(D_t)D_t$ is larger than γ with high probability, MPR may become negative, because the bond prices become so countercyclical that the investors are eager to buy stock for hedging purposes, even if the stock offers a negative excess return. Furthermore, if $r'(D_t)D_t$ is decreasing in D_t, MPR becomes procyclical.

By Proposition 6.3, countercyclical macroeconomic uncertainty together

[30] In fact, we show that the following is true: for any interval $[d_1, d_2] \subset \mathbb{R}_+$ there exists an $\epsilon > 0$ such that λ_t is monotone decreasing in D_t in the interval, as long as $r' < \epsilon$ and $|r''| < \epsilon$.

with countercycical risk aversion generate countercyclical dynamics of MPR. The latter dynamics, in turn, determine the sign of nonmyopic volatility, , as can be seen from expression (13) and the definition of $\lambda(t, T)$. More precisely, we arrive at the following

Proposition 6.4. *Under Assumption 6.1, the nonmyopic volatility is positive if* $\gamma^U \geq 1$ *and* r' *is sufficiently small.*

The reasoning behind this result is as follows. Under Assumption 6.1, MPR is low in good states, but the future aggregate risk aversion is also (relatively) low and the expected future dividends are high. Therefore, the agent is willing to hold the stock despite low MPR. This makes the price go up in good states and, by similar arguments, go down in bad states, driving the nonmyopic part of the price volatility up. Put differently, the tension between the movements in future dividend and MPR creates excess volatility, because larger the dividend, larger the demand for the stock, while at the same time lower the MPR, lower the demand. However, if the interest rate is highly procyclical, discounted dividends $e^{-\int_t^T r_s ds} D_T$ may exhibit countercyclical behavior, discounted dividends which, by (13), gives rise to a negative excess volatility.

One important consequence of Proposition 6.4 and Theorem 4.1 is that, under their assumptions, the following *risk-return inequality* holds:

$$\lambda_t \;<\; \sigma_t^S \max_x \gamma^U(x)\,.$$

The reason is that MPR is driven by the myopic volatility, which is smaller than the stock volatility by Theorem 4.2 and Proposition 6.4. Due to its myopic nature, equilibrium MPR "underestimates" the true stock price volatility and is unable to account for future stock price and return fluctuations. Since MPR is countercyclical, the stock is cheap in bad states (those with low D_t) and offers a high instantaneous return. This motivates the agents to buy more shares in those states, and the stock becomes more volatile, which, in turn, stabilizes the demand. A similar argument applies in good states.

We now study the implications of MPR countercyclicality for nonmyopic equilibrium optimal portfolios. Recall that the absolute prudence of agent k is defined by

$$P_k(x) \; = \; -\frac{u_k'''(x)}{u_k''(x)}$$

and absolute risk tolerance is

$$R_k(x) \; = \; \frac{x}{\gamma_k(x)}$$

Then, we have

Theorem 6.1. *Under Assumption 6.1, and supposing that r' is not too large, the MPR hedging component $\pi_t^{\text{MPR hedge}}$ is positive if and only if the product of prudence and risk tolerance is below two*[31]*, that is*

$$P_k(x)\, R_k(x) \; \le \; 2 \qquad \text{for all } x.$$

The result of Theorem 6.1 is somewhat unexpected at first glance. Since the optimal portfolio of a log investor is always myopic, one would expect that the sign of the hedging component only depends on whether risk aversion is above or below one (as it does for the IR hedging component). However, Theorem 6.1 shows that the hedging motive depends on properties of *three* derivatives of the utility function.

We now analyze in more detail the reasons behind this result, showing that it is really the sensitivity of the risk tolerance that drives it. Let for

[31] Note that the product of prudence and risk tolerance is exactly two for the log investor, for whom, the hedging portfolio is zero.

simplicity r be constant and recall expression (19):

$$e^{r(T-t)}\,\sigma_t^S\,W_{k\,t}\,\pi_{k\,t}^{\text{MPR hedge}} \;=\; -\text{Cov}_t^Q\big(\lambda(t\,,\,T)\,,\,(W_{k\,T}\;-\;R_{k\,T})\big)$$

By Assumption 6.1, $\lambda(t\,,\,T)$ is countercyclical. Since markets are complete, all terminal wealth amounts $W_{k\,T}$ are co-monotone, increasing in D_T and, consequently, the wealth hedge $-\text{Cov}_t^Q(\lambda(t\,,\,T)\,,\,W_{k\,T})$ is positive. The size of the risk tolerance hedge depends on the cyclical properties of agent's risk tolerance $R_{k\,T}$. A direct calculation shows that

$$\frac{d}{dx}\,R_k(x) \;=\; -1 \;+\; P_k(x)\,R_k(x).$$

If the product $P_k\,R_k$ is large, the risk tolerance $R_{k\,T}$ is highly sensitive to wealth fluctuations and, consequently, highly procyclical. Then, the risk tolerance hedge

$$\text{Cov}_t^Q\big(\lambda(t\,,\,T)\,,\,R_{k\,T})\big)$$

is very negative, dominates over the wealth hedge

$$-\,\text{Cov}_t^Q\big(\lambda(t\,,\,T)\,,\,W_{k\,T})\big)$$

and the MPR hedge is negative. By contrast, if $P_k\,R_k$ is small, the risk tolerance hedge is not very negative. Therefore, the wealth hedge dominates and the MPR hedge is positive. The threshold of two appears because $W_{k\,T} - R_k(W_{k\,T})$ is monotone increasing in $W_{k\,T}$ if and only if $P_k\,R_k\;\le 2$.

The appearance of prudence is also related to precautionary savings. As Kimball (1990) showed in a static, one period model, the strength of the precautionary savings motive for an agent anticipating stochastic fluctuations in his future income is determined by his prudence P_k. Here, P_k plays a similar role, determining the strength of savings/investment motive for an agent, anticipating future changes in the stochastic investment opportunity set.

If we consider the Arrow (1965) hypothesis that $\gamma_k(x)$ is increasing, a direct calculation shows that it holds if and only if $x\, P_k(x) \;\le\; \gamma_k(x) \,+\, 1\,.$ If $\gamma_k(x) \ge 1$, we get $P_k(x)\, R_k(x) \;=\; x\, P_k(x)/\gamma_k(x) \;\le\; 2$. Theorem 6.1 then implies the following

Corollary 6.1. *Suppose that* $\gamma_k(x) \;\ge\; 1$ *and is increasing and* r' *is not too large. Then, under Assumption 6.1, the MPR hedge is positive.*

For the benchmark CRRA utility, $P_k\, R_k \;=\; 1 + \gamma_k^{-1}$ and the results take a simpler form

Corollary 6.2. *Suppose that* $\gamma_k = \text{const}$, r' *is not too large, and Assumption 6.1 holds. Then, the MPR hedge is positive if and only if* $\gamma_k \;\ge\; 1\,.$

The intuition behind Corollary 6.2 is as follows. The marginal utility $u_k'(x) \;=\; x^{-\gamma_k}$ of an agent with high risk aversion $\gamma_k > 1$ is high in bad states (low D_t). Therefore, any additional unit of consumption in those states is highly valuable for him. Since MPR is countercyclical, it is high in bad states. This makes the stock a highly attractive instrument for agent k to hedge against those states and makes him buy additional shares. On the other hand, an agent with low risk aversion $\gamma_k < 1$ is not "afraid" of the bad states and bets on the realization of good states with high D_t. Since MPR is low in those states, it is optimal for him to sell some of his stock holdings, creating a negative hedging demand.

We complete our discussion of optimal portfolios with a monotonicity result. We will need the following

Definition 6.2 (Ross (1981)). *Agent k is more risk averse than agent j in the sense of Ross, if*

$$\inf_x \gamma_k(x) \geq \sup_x \gamma_j(x).$$

In this case we write $\gamma_k \geq_R \gamma_j$.

This definition was introduced by Ross (1981) in the context of a static, one period problem with two risky assets. Ross showed that the optimal investment in the riskier asset is generally not monotone in risk aversion, if we only require a weak, pointwise inequality in risk aversion. The reason is that the optimal portfolio choice with wealth dependent risk aversion and two risky assets becomes a non-local problem and the local properties of risk aversion are not sufficient for the analysis. A similar phenomenon arises in our dynamic, multi-period optimization: even though one of the assets is locally riskless, the amount of money invested into it changes over time and thus, effectively, we get a problem with two risky assets, and Definition 6.2 becomes the right concept to consider.

First, we note that the following is true.

Proposition 6.5. *If an agent is less (more) risk averse than the log agent then he invests more (less) into the risky asset then the log agent.*[32]

Proposition 6.5 is very general and holds for any economy. To compare portfolios of two non-logarithmic agents, we will need to make additional assumptions. We have

Proposition 6.6. *Suppose that risk aversions of all agents are above one. Then, under Assumption 6.1, the optimal portfolios are monotone decreasing in risk aversion in the sense of Ross if r' is not too large.*

Monotonicity of optimal portfolios is important – almost all papers on heterogeneous equilibria use this monotonicity property as the basis for eco-

[32]The proof follows directly from Proposition B.1 in the appendix.

nomic intuition. See, e.g., Dumas (1989), Wang (1996), Basak and Cuoco (1998), Basak (2005), Bhamra and Uppal (2009a). However, to the best of our knowledge, no proof of this property has ever been given even in an economy with only two agents.

7 Example: Mean-Reverting Dividends and CRRA agents

We now illustrate the results from the previous sections on our benchmark example. That is, we assume

Assumption 7.1. *The log dividend $A_t = \log D_t$ follows an autoregressive process*

$$dA_t = (a - bA_t)dt + \sigma\, dB_t,$$

interest rate r is zero, and all agents in the economy have CRRA preferences, $\gamma_k(x) = \gamma_k$.

Under Assumption 7.1, $\sigma^D = \sigma$, $c = b$. Consequently, all the formulae substantially simplify and all the dynamic properties of equilibrium are determined solely by the properties of the aggregate risk aversion γ^U. The following proposition summarizes our main results.

Proposition 7.1. *Under Assumption 7.1,*

(1) the market price of risk

$$\lambda_t = E_t^Q[\,\gamma^U(D_T)\,]\,e^{-b\,(T-t)}\,\sigma$$

is counter-cyclical;

(2) price volatility is given by

$$\sigma_t^S = \sigma\, e^{-b\,(T-t)} + \sigma_t^{\text{nonmyopic}}$$

where the nonmyopic (excess) volatility

$$\sigma_t^{\text{nonmyopic}} \;=\; -\frac{e^{-b\,(T-t)}\,\sigma}{S_t}\,\text{Cov}_t^Q(\gamma^U(D_T)\,,\,D_T)$$

is always positive;

(3) optimal portfolios are monotone decreasing in risk aversion for risk aversions above 1. *They are given by*

$$\pi_{k\,t} \;=\; \frac{\lambda_t}{\gamma_k\,\sigma_t^S} \;+\; \pi_{k\,t}^{\text{hedging}}$$

where the hedging component

$$\pi_{k\,t}^{\text{hedging}} \;=\; \frac{1}{W_{k\,t}}\,(\gamma_k^{-1}\;-\;1)\,\frac{e^{-b\,(T-t)}\,\sigma}{\sigma_t^S}\,\text{Cov}_t^Q(\gamma^U(D_T)\,,\,W_{k\,T})$$

is positive if and only if $\gamma_k \;>\; 1$.

It is interesting to note that the hedging component depends on risk aversion in an asymmetric way. We have

$$W_{k\,T} \;=\; (\xi_T\,y_k)^{-\gamma_k^{-1}}\,.$$

Therefore, the sensitivity of wealth to changes in the state variable D_T is larger for the agents with small risk aversion. For example, , for the agents with very small risk aversion, the coefficient $\gamma_k^{-1} - 1$ controlling the size of the hedging demand is be very large, and their hedging demand will be very negative. By contrast, for large risk aversion, both effects are small and therefore their hedging demand is positive but relatively small.

We conclude this section with another interesting result related to heterogeneity in risk aversions, showing how the size of the risk-return tradeoff, the size of the ratio $\sigma_t^S/\sigma_t^{\text{myopic}}$ and the size of the equilibrium optimal portfolios depend on the magnitude of the heterogeneity. Let

$$\gamma \stackrel{def}{=} \min_k \gamma_k \ , \ \Gamma \stackrel{def}{=} \max_k \gamma_k .$$

Proposition 7.2. *The following is true:*

(1) the market price of risk satisfies

$$\gamma \, \sigma \, e^{-b(T-t)} \ \leq \ \lambda_t \ \leq \ \Gamma \, \sigma \, e^{-b\,(T-t)}$$

(2) the stock volatility satisfies

$$\sigma \, e^{-b(T-t)} \ \leq \ \sigma_t^S \ \leq \ \sigma \, e^{-b(T-t)} \left(1 \ + \ \Gamma \ - \ \gamma\right)$$

(3) the risk-return tradeoff satisfies

$$\frac{\gamma}{1 \, + \, \Gamma \, - \, \gamma} \ \leq \ \frac{\lambda_t}{\sigma_t^S} \ \leq \ \Gamma \tag{20}$$

(4) optimal portfolios satisfy

$$\begin{aligned} \frac{1}{\gamma_k} \frac{\gamma}{1 \, + \, \Gamma \, - \, \gamma} \ \leq \ \ \pi_{k\,t} \ \ \leq \ \frac{1}{\gamma_k} \Big(\Gamma \ + \ (\gamma_k - 1)\,(\Gamma \ - \ \gamma) \Big) \quad & \text{if } \gamma_k > 1 \\ \frac{1}{\gamma_k} \frac{\gamma}{1 \, + \, \Gamma \, - \, \gamma} \ \leq \ \ \pi_{k\,t} \ \ \leq \ \frac{1}{\gamma_k} \Gamma \quad & \text{if } \gamma_k < 1 \, . \end{aligned}$$

These results imply that the size $\Gamma - \gamma$ of heterogeneity plays a crucial role for determining the size of excess volatility, risk-return tradeoff and the size of equilibrium optimal portfolios. Using the terminology of Dumas, Kurshev and Uppal (2009), investors are taking advantage of heterogeneous risk attitudes in the economy, which generates excess volatility and stochastic fluctuations of the market price of risk.

Bounds (20) and (21) are especially useful for empirical analysis because they do not depend on the parameters of the dividend process. For example, finding the empirical range

$$\left[\min \frac{\lambda_t}{\sigma_t^S}\ ,\ \max \frac{\lambda_t}{\sigma_t^S}\right]$$

immediately allows us to determine the *size of heterogeneity* in the economy. Moreover, given the estimates for γ and Γ from risk-return tradeoffs, the data on empirical portfolio holdings together with bounds (21) could be used to study the cross-sectional distribution of risk aversions in the economy.

8 Conclusions

We have obtained general representations of the equilibrium market price of risk, stock volatility, as well as of the optimal portfolios, in terms of expected values and covariances under the risk-neutral measure. The equilibrium values depend on the aggregate relative risk aversion and dividend volatility, discounted at a specific rate, called the rate of macroeconomic fluctuations. These results are universal in the sense that they hold in any exchange economy with an arbitrary Markov dividend, arbitrary stochastic interest rates and arbitrary rational agents, maximizing utility from terminal consumption.

Our results provide several of theoretical predictions that can be tested empirically: (1) the drift of the market price of risk is equal the product of MPR and its volatility plus the rate of macroeconomic fluctuations; (2) market price of risk is driven by the myopic volatility; (3) stock volatility consists of myopic and non-myopic volatility components, and both volatilities can be calculated empirically given the empirical dividend volatility, rate of macroeconomic fluctuations and the empirical pricing kernel; (4) both the optimal myopic and the optimal non-myopic portfolios can be calculated given the agent's desired terminal wealth profile and the empirical quantities from (3).

Furthermore, our results highlight general mechanisms behind the phenomena of MPR countercyclicality, excess volatility and nonmyopic optimal portfolios. In particular, we show that excess volatility is determined by the interplay between the market price of risk and the interest rate cyclicality, that the size and the sign of the interest rate hedge is determined by interest rate volatility and the size of agent's risk aversion, and that the MPR hedge consists of two components, the wealth hedge and the risk tolerance hedge, whose relative size is determined by the interplay between prudence and risk tolerance.

We believe that these new insights can be useful both in theory for testing various versions of CCAPM, and in practice for determining non-myopic optimal portfolios.

Appendix

A Proofs: Equilibrium Price Dynamics

Denote by $\mathcal{D}_t$ the Malliavin derivative operator. [33] The following is the main technical result of the paper.

[33] For an expedient introduction to Malliavin derivatives see Detemple, Garcia and Rindisbacher (2003).

Proposition A.1. *The drift and volatility of the stock price are given by*

$$
\begin{aligned}
\mu_t^S &= r(D_t) + \sigma_t^S \Bigg(E_t^Q \left[\gamma^U(D_T)\,(\mathcal{D}_t D_T)/D_T\right] \\
&\quad - \int_t^T E_t^Q \left[r'(D_s)\,(\mathcal{D}_t D_s)\right] ds \Bigg) \\
\sigma_t^S &= \frac{1}{E_t[\xi_T\, D_T]} E_t[(1-\gamma_U(D_T)\xi_T \mathcal{D}_t D_T)] \\
&\quad - \frac{E_t\left[e^{\int_0^T r(D_s)ds}\left(\int_t^T r'(D_s)\,(\mathcal{D}_t D_s)\,ds\right)\xi_T\right] + E_t\left[e^{\int_0^T r(D_s)ds}\,\mathcal{D}_t \xi_T\right]}{E_t\left[e^{\int_0^T r(D_s)ds}\,\xi_T\right]}
\end{aligned}
\tag{21}
$$

and the optimal portfolio of agent k *is given by*

$$
\begin{aligned}
\pi_{k\,t} &= \frac{1}{\sigma_t^S} \frac{E_t\left[\mathcal{D}_t\,\xi_T\,(y_k\,\xi_T\,I_k'(y_k\,\xi_T) + I_k(y_k\,\xi_T))\right]}{E_t[\xi_T\,I_k(y_k\,\xi_T)]} \\
&\quad - \frac{E_t\left[e^{\int_0^T r(D_s)ds}\left(\int_t^T r'(D_s)\,(\mathcal{D}_t D_s)\,ds\right)\xi_T\right] + E_t\left[e^{\int_0^T r(D_s)ds}\,\mathcal{D}_t \xi_T\right]}{E_t\left[e^{\int_0^T r(D_s)ds}\,\xi_T\right]}
\end{aligned}
\tag{22}
$$

where

$$
\mathcal{D}_t D_T = D_t \sigma^D(D_t) e^{\delta_T - \delta_t} \tag{23}
$$

with

$$
\begin{aligned}
\delta_t &= \int_0^t [D_s(\mu^D)'(D_s) - 0.5(D_s(\sigma^D)'(D_s))^2 - D_s(\sigma^D)'(D_s)\sigma^D(D_s)]ds \\
&\quad + \int_0^t D_s(\sigma^D)'(D_s)dB_s
\end{aligned}
\tag{24}
$$

and

$$
\mathcal{D}_t \xi_T = -\frac{1}{D_T}\gamma^U(D_T)\xi_T \mathcal{D}_t D_T\,.
$$

Proof of Proposition A.1. By definition,

$$
\xi_t = e^{-\int_0^t r(D_s)ds} M_t = e^{-\int_0^t r(D_s)ds} E_t[M_T] = E_t[e^{\int_t^T r(D_s)ds}\,\xi_T]
$$

The price S_t and the wealth of agent k satisfy

$$\log S_t \;=\; \int_0^t r(D_s)\,ds \;+\; \log E_t[\xi_T\, D_T] - \log E_t[e^{\int_0^T r(D_s)ds}\,\xi_T]$$

and

$$\log W_{k\,t} \;=\; \int_0^t r(D_s)\,ds \;+\; \log E_t\left[\xi_T\, I_k(y_k\,\xi_T)\right] \;-\; \log E_t[e^{\int_0^T r(D_s)ds}\,\xi_T]\,.$$

We get the volatility σ_t^S as the Malliavin derivative $\mathcal{D}_t \log S_t$ and we get $\sigma_t^S \pi_{k\,t}$ as the Malliavin derivative $\mathcal{D}_t \log W_{k\,t}$. Thus, we have

$$\pi_{k\,t} = \frac{\mathcal{D}_t \log W_{k\,t}}{\mathcal{D}_t \log S_t}\,. \tag{25}$$

We will now calculate the Malliavin derivatives. For process D, it is well known that the Malliavin derivative

$$Y_t := \mathcal{D}_t D_u\ ,\quad u \ge t$$

satisfies the linear SDE

$$dY_u = (D_u(\mu^D)'(D_u) + \mu^D(D_u))Y_u du + (D_u(\sigma^D)'(D_u) + \sigma^D(D_u))Y_u dB_u\ ,\ \ u \ge t$$

$$Y_t = D_t \sigma^D(D_t)$$

and (23) follows. Using this and (7), we can compute

$$\mathcal{D}_t \xi_T \;=\; U''(D_T)\mathcal{D}_t D_T\ . \tag{26}$$

Using the identity

$$\mathcal{D}_t E_t[X] = E_t[\mathcal{D}_t X]$$

we can compute

$$
\begin{aligned}
&\mathcal{D}_t \log W_{k\,t} \\
&\quad = \frac{1}{E_t[\xi_T I_k(y_k\,\xi_T)]} E_t\left[y_k \xi_T I_k'(y_k\,\xi_T)\mathcal{D}_t\xi_T + I_k(y_k\,\xi_T)\mathcal{D}_t\xi_T\right] \\
&- \frac{E_t\left[e^{\int_0^T r(D_s)ds}\left(\int_t^T r'(D_s)\,(\mathcal{D}_t D_s)\,ds\right)\xi_T\right] \;+\; E_t\left[e^{\int_0^T r(D_s)ds}\,\mathcal{D}_t\xi_T\right]}{E_t\left[e^{\int_0^T r(D_s)ds}\,\xi_T\right]}
\end{aligned} \tag{27}
$$

and

$$
\begin{aligned}
\mathcal{D}_t \log S_t &= \frac{1}{E_t[\xi_T\,D_T]} E_t[D_T \mathcal{D}_t\xi_T + \xi_T \mathcal{D}_t D_T] \\
&- \frac{E_t\left[e^{\int_0^T r(D_s)ds}\left(\int_t^T r'(D_s)\,(\mathcal{D}_t D_s)\,ds\right)\xi_T\right] \;+\; E_t\left[e^{\int_0^T r(D_s)ds}\,\mathcal{D}_t\xi_T\right]}{E_t\left[e^{\int_0^T r(D_s)ds}\,\xi_T\right]}.
\end{aligned} \tag{28}
$$

It remains to show the expression for the drift. By the martingale property, we can write,

$$
\frac{dE_t[\xi_T\,D_T]}{E_t[\xi_T\,D_T]} \;=\; U_t dW_t\;,\;\; \frac{dE_t[e^{\int_0^T r(D_s)ds}\,\xi_T]}{E_t[e^{\int_0^T r(D_s)ds}\,\xi_T]} \;=\; V_t dW_t
$$

where, by Clarke-Ocone formula and (26),

$$
U_t \;=\; \frac{\mathcal{D}_t E_t[\xi_T\,D_T]}{E_t[\xi_T\,D_T]} \;=\; \frac{1}{E_t[\xi_T\,D_T]} E[\xi_T(1-\gamma_U(D_T))\mathcal{D}_t D_T]
$$

and

$$
\begin{aligned}
V_t \;&=\; \frac{\mathcal{D}_t E_t[e^{\int_0^T r(D_s)ds}\xi_T]}{E_t[e^{\int_0^T r(D_s)ds}\xi_T]} \\
=\; -&\frac{E_t[e^{\int_0^T r(D_s)ds}\,(\gamma^U(D_T)\xi_T\mathcal{D}_t D_T)/D_T] \;-\; E_t\left[e^{\int_0^T r(D_s)ds}\left(\int_t^T r'(D_s)\,(\mathcal{D}_t D_s)\,ds\right)\xi_T\right]}{E_t[e^{\int_0^T r(D_s)ds}\,\xi_T]}.
\end{aligned} \tag{29}
$$

Applying Ito's formula, we get

$$d\log S_t \;=\; r(D_t)\,dt + d\log\frac{E_t[\xi_T\,D_T]}{E_t[\xi_T]} \;=\; \frac{1}{2}(2\,r(D_t) + V_t^2 - U_t^2)dt + (U_t - V_t)dW_t.$$

Therefore,

$$\mu_t^S \;=\; r(D_t) \;+\; \frac{1}{2}(V_t^2 - U_t^2 + (U_t - V_t)^2) \;=\; r(D_t) \;+\; V_t\,(V_t - U_t)$$

and thus, by (28),

$$\begin{aligned}\mu_t^S \;&=\; r(D_t)\\ &+ \frac{E_t[e^{\int_0^T r(D_s)ds}(\gamma^U(D_T)\xi_T\mathcal{D}_tD_T)/D_T] \;-\; E_t\left[e^{\int_0^T r(D_s)ds}\,\left(\int_t^T r'(D_s)\,(\mathcal{D}_tD_s)\,ds\right)\,\xi_T\right]}{E_t[e^{\int_0^T r(D_s)ds}\,\xi_T]}\\ &\times \mathcal{D}_t\log S_t\\ &= r(D_t) + \sigma_t^S\,\left(E_t^Q\left[\gamma^U(D_T)\,(\mathcal{D}_tD_T)/D_T\right] \;-\; \int_t^T\; E_t^Q\left[r'(D_s)\,(\mathcal{D}_tD_s)\right]\,ds\right)\end{aligned} \tag{30}$$

Q.E.D.

The following result allows us to rewrite the Malliavin derivative $\mathcal{D}_tD$ without involving stochastic integrals. It has also been proved by Detemple, Garcia and Rindisbacher (2003) in a slightly different form, but we present a derivation here for the reader's convenience.

Lemma A.1. *We have*

$$\mathcal{D}_tD_T \;=\; D_T\,\sigma^D(D_T)\,e^{-\int_t^T\,\rho_s\,ds}\,. \tag{31}$$

Proof. By Ito's formula,

$$\log(D_T\,\sigma^D(D_T))\;-\;\log(D_t\,\sigma^D(D_t))\;=\;\int_t^T(\sigma^D(D_s)+D_s\,(\sigma^D)'(D_s))\,dB_s$$

$$+\;\int_t^T\big((\sigma^D(D_s)\;+D_s\,(\sigma^D)'(D_s))\,\sigma^D(D_s)^{-1}\,\mu^D(D_s)\big)\,ds$$

$$+\;\frac{1}{2}\int_t^T\Big((2(\sigma^D)'(D_s)+D_s(\sigma^D)''(D_s))\,\sigma^D(D_s)$$

$$-\,(\sigma^D(D_s)\;+(D_s\,(\sigma^D)'(D_s))^2)\Big)\,ds. \quad (32)$$

It remains to show that

$$\rho_s\;=\;c(D_s)$$

where

$$c(x)\;=\;-x\,(\mu^D)'(x)\;+\;x\,(\sigma^D)'(x)\,\sigma^D(x)^{-1}\,\mu^D(x)$$

$$+\;(\sigma^D)'(x)\,\sigma^D(x)\,x\;+\;0.5\,(\sigma^D)''(x)\,x^2\,\sigma^D(x). \quad (33)$$

This claim can be verified by direct calculation. Q.E.D.

Proof of Theorem 4.1. The proof follows directly by substituting (31) into (21). Q.E.D.

We will need the following known

Lemma A.2. *For any one-dimensonal diffusion, the function*

$$G(t\,,\,x)\;=\;E[g(D_T)\,|\,D_t\;=\;x]$$

is monotone increasing (decreasing) in x for all $t\in[0,T]$ if and only if so does $g(x)$. Furthermore, if both $g(x)$ and $h(x)$ are increasing (or both decreasing), then

$$E_t[g(D_T)]\,E_t[h(D_T)]\;\le\;E_t[g(D_T)\,h(D_T)].$$

If both g,h are strictly increasing (or both strictly decreasing), then the inequality is also strict unless D_T is constant almost surely. If one function is increasing and the other is decreasing, then the inequality reverses.

Proof. See Herbst and Pitt (1991). Q.E.D.

Lemma A.3. *Suppose that F and $G_1, \cdots, G_N$ are monotone increasing functions. Then, for any $N \in \mathbb{N}$ and any $\{t_1 \le \cdots t_N\} \subset [t, T]$,*

$$E[F(X_T)\, G_1(X_{t_1}) \cdots G_N(X_{t_N}) \,|\, X_t = x]$$

is monotone increasing in x and

$$E_t[F(X_T)\, G_1(X_{t_1}) \cdots G_N(X_{t_N})] \;\ge\; E_t\,[F(X_T)\,]\; E_t\,[\,G_1(X_{t_1}) \cdots G_N(X_{t_N})\,]\;.$$

Proof. The proof is by induction. For $N = 1$, we have

$$E_t[F(X_T)\, G_{t_1}(X_{t_1})] \;=\; E_t[\, E_{t_1}[F(X_T)]\, G_{t_1}(X_{t_1})].$$

By Lemma A.2, the function inside the expectation is increasing in X_{t_1} and another application of Lemma A.2 provides monotonicity of $E_t[F(X_T)\, G_{t_1}(X_{t_1})]$. Now, by Lemma A.2,

$$\begin{aligned} E_t[\, E_{t_1}[F(X_T)]\, G_1(X_{t_1})] \;&\ge\; E_t[\, E_{t_1}[F(X_T)]\,]\, E_t[G_{t_1}(X_{t_1})] \\ &=\; E_t[F(X_T)]]\, E_t[G_{t_1}(X_{t_1})] \qquad (34) \end{aligned}$$

and we are done. Suppose now that the claim has been proved for N. Then,

$$\begin{aligned} &E_t[F(X_T)\, G_1(X_{t_1}) \cdots G_N(X_{t_N})] \\ &\quad=\; E_t[\, G_1(X_{t_1})\, E_{t_1}[\, F(X_T)\, G_2(X_{t_2}) \cdots G_N(X_{t_N})]\,] \\ &\quad\ge\; E_t[E_{t_1}[\, F(X_T)\,]\, E_{t_1}[G_1(X_{t_1})\, G_2(X_{t_2}) \cdots G_N(X_{t_N})]\,] \qquad (35) \end{aligned}$$

and the claim follows from Lemma A.2 and the induction hypothesis. Q.E.D.

Lemma A.4. *Suppose that f and g are both increasing (decreasing). Then, the following is true*

(1) the function[34]

$$E\left[f(D_T)\, e^{\int_t^T g(D_s)\, ds} \,|\, D_t \;=\; x \right]$$

[34] Item (1) of Lemma A.4 is contained in Mele (2007).

is also monotone increasing (decreasing);

(2) *we have*

$$\mathrm{Cov}_t\left(h(D_T)\,,\,f(D_T)\,e^{\int_t^T g(D_s)\,ds}\right)\;\geq\;0$$

if h has the same direction of monotonicity as f and g, and the inequality reverses if h has the opposite direction of monotonicity.

Proof of Lemma A.4.. The claim follows from Lemma A.3, approximating the integral $\int_t^T$ by discrete integral sums. Q.E.D.

Proof of Theorem 4.2. The proof follows directly from Proposition A.1 and (31). The fact that $\sigma_t^S > 0$ when r' is sufficiently small follows from (2), Lemma A.4 and

$$\sigma_t^S \;=\; \frac{\partial}{\partial D_t}\,S(t\,,\,D_t)\,\sigma^D(D_t)\,.$$

Q.E.D.

B Proofs: Optimal Portfolios

We start with the following auxiliary

Proposition B.1. *The optimal portfolio $\pi_{k\,t}$ is given by*

$$\sigma_t^S\,\pi_{k\,t}\;=\;\frac{E_t^Q\left[\lambda(t,T)\,e^{-\int_t^T r(D_s)ds}\,W_{k\,T}\,\left(\gamma_k^{-1}(W_{k\,T})\;-\;1\right)\right]}{W_{kt}}\;+\;\lambda_t \qquad (36)$$

Proof of Proposition B.1. The claim follows directly from Proposition A.1 and (31). Q.E.D.

Proof of Proposition 6.6. By (36), we need to show that

$$\frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T \rho_s ds}\,W_{kT}\,\left(1-\gamma_k^{-1}(W_{k\,T})\,\right)\right]}{E_t^Q[W_{kT}]}$$

$$\geq\;\frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T \rho_s\,ds}\,W_{jT}\,\left(1-\gamma_j^{-1}(W_{j\,T})\right)\right]}{E_t^Q[W_{j\,T}]}\,. \qquad (37)$$

We only prove case (1). Case (2) is analogous. Since, by assumption,

$$\inf\,(1 - \gamma_{k\,T}^{-1}) \;\geq\; \sup\,(1 - \gamma_{j\,T}^{-1}),$$

it suffices to show that

$$\frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T \rho_s ds}\,W_{kT}\right]}{E_t^Q[W_{kT}]} \;\geq\; \frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T \rho_s ds}\,W_{j\,T}\right]}{E_t^Q[W_{jT}]}\,.$$

Introduce a new probability measure

$$dQ^k \;=\; \frac{W_{k\,T}}{E^Q[W_{k\,T}]}\,dQ$$

and let

$$f(x) \;=\; \frac{I_j(y_j x)}{I_k(y_k\,x)}.$$

and $z_i \;=\; I_i(\lambda_i\,x)\,,\; i\,\in\{j\,,\,k\}$. Then,

$$\begin{aligned} f'(x) \;&=\; \frac{I_j(y_j\,x)}{x\,I_k(y_k\,x)}\left(y_j\,x\,\frac{I_j'(y_j\,x)}{I_j(y_j\,x)} \;-\; y_k\,x\,\frac{I_k'(y_j\,x)}{I_k(y_k\,x)}\right) \\ &=\; \frac{I_j(y_j\,x)}{x\,I_k(y_k\,x)}\left(\frac{u_j'(z_1)}{z_1\,u_j''(z_1)} \;-\; \frac{u_k'(z_2)}{z_2\,u_k''(z_2)}\right) \\ &=\; \frac{z_1}{x\,z_2}\,(\gamma_k^{-1}(z_2) \;-\; \gamma_j^{-1}(z_1)) \;\leq\; 0, \end{aligned} \qquad (38)$$

that is, f is decreasing. Therefore, $f(U'(D_T))$ is increasing and, by Lemma A.4,

$$\begin{aligned} &\frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T \rho_s ds}\,W_{j\,T}\right]}{E_t^Q[W_{jT}]} \\ &\qquad=\; \frac{E_t^{Q_k}\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T \rho_s ds}\,f(U'(D_T))\right]}{E_t^{Q_k}[f(U'(D_T))]} \\ &\leq\; E_t^{Q_k}\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T \rho_s ds}\right] \;=\; \frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(D_T)\,e^{-\int_t^T \rho_s ds}\,W_{kT}\right]}{E_t^Q[W_{kT}]} \end{aligned} \qquad (39)$$

Q.E.D.

Proof of Proposition 5.1. Let for simplicity $t = 0$. By definition, the value function is

$$U_k(x) \;=\; E[\, u_k(I_k(\xi_T\, y_k))\,]$$

where $y_k = y_k(x)$ solves

$$x \;=\; E[\xi_T\, I_k(\xi_T\, y_k)].$$

Differentiating this identity, we get

$$y_k' \;=\; E[\xi_T^2\, I_k'(\xi_T\, y_k)]^{-1}$$

and therefore

$$U_k'(x) \;=\; E\,[u_k'(I_k(\xi_T\, y_k))\, I_k'(\xi_T\, y_k)\, \xi_T\,]\, \lambda')k \;=\; E\,[\,\xi_T\, y_k\, I_k'(\xi_T\, y_k)\, \xi_T\,]\, y_k' \;=\; y_k.$$

Consequently,

$$U_k''(x) \;=\; y_k' \;=\; \frac{1}{E[\xi_T^2\, I_k'(\xi_T y_k)]}$$

and

$$\gamma_{k0}(x) = = \;-\frac{x}{y_k\, E[\xi_T^2\, I_k'(\xi_T y_k)]} \;=\; -\frac{E[\xi_T\, I_k(\xi_T y_k)]}{E[\xi_T^2\, y_k\, I_k'(\xi_T y_k)]}\,.$$

Differentiating the identity

$$u_k'(I_k(x)) \;=\; x$$

we get

$$I_k'(x) \;=\; (u_k''(x))^{-1}$$

and therefore

$$y_k \xi_T\, I_k'(y_k \xi_T) \;=\; -\,\gamma_{kT}^{-1} W_{k\,T}.$$

Q.E.D.

Proof of Theorem 5.1. By Propositions B.1 and 5.1,

$$\begin{aligned}
\sigma_t^S\,\pi_{k\,t}^{\text{hedging}} \;&=\; \sigma_t^S\,\left(\pi_{k\,t}\;-\;\pi_{k\,t}^{\text{myopic}}\right) \\
&= \frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(t\,,\,T)\,e^{-\int_t^T r(D_s)ds}\,W_{k\,T}\,\left(\gamma_{kT}^{-1}\;-\;1\right)\right]}{E_t^Q[e^{-\int_t^T r(D_s)ds}\,W_{kT}]}\;+\;\lambda_t \\
&\qquad -\;\lambda_t\,\frac{E_t^Q[e^{-\int_t^T r(D_s)ds}\,\gamma_{kT}^{-1}\,W_{k\,T}]}{E_t^Q[e^{-\int_t^T r(D_s)ds}\,W_{k\,T}]} \\
&= \frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(t\,,\,T)\,e^{-\int_t^T r(D_s)ds}\,W_{k\,T}\,\left(\gamma_{kT}^{-1}\;-\;1\right)\right]}{E_t^Q[e^{-\int_t^T r(D_s)ds}\,W_{kT}]} \\
&\qquad -\;\lambda_t\,\frac{E_t^Q[e^{-\int_t^T r(D_s)ds}\,(\gamma_{kT}^{-1}-1)\,W_{kT}]}{E_t^Q[e^{-\int_t^T r(D_s)ds}\,W_{kT}]} \\
&= \frac{1}{E_t^Q[W_{k\,T}]}\Big(E_t^Q\left[\gamma^U(D_T)\,\sigma^D(t\,,\,T)\,e^{-\int_t^T r(D_s)ds}\,W_{k\,T}\,\left(\gamma_{kT}^{-1}\;-\;1\right)\right] \\
&\quad -\;E_t^Q[\gamma^U(D_T)\,\sigma^D(t\,,\,T)]\,E_t^Q[(\gamma_{kT}^{-1}\;-\;1)\,e^{-\int_t^T r(D_s)ds}\,W_{k\,T}]\Big) \\
&\qquad +\;\frac{E_t^Q[e^{-\int_t^T r(D_s)ds}\,(\gamma_{kT}^{-1}-1)\,W_{kT}]}{E_t^Q[e^{-\int_t^T r(D_s)ds}\,W_{kT}]}\,E_t^Q\left[\int_t^T\,\sigma^r(t,\tau)\,d\tau\right] \qquad (40)
\end{aligned}$$

which is what had to be proved. Q.E.D.

Proofs of Proposition 6.3, Proposition 6.4 and Theorem 6.1. Propositions 6.3, 6.4 follows directly from Lemma A.4. Theorem 6.1 follows from Theorem 5.1 and Lemma A.4 since

$$f(x)\;=\;x\;-\;R_k(x)$$

is increasing if and only if

$$f'(x)\;=\;1\;+\;\frac{(u_k'')^2\;-\;u_k'(x)\,u_k'''(x)}{(u_k''(x))^2}\;=\;\frac{1}{(u_k''(x))^2}\,(2\;-\;P_k(x)\,R_k(x))\;\geq\;0$$

and is decreasing otherwise. Q.E.D.

Proof of Proposition 7.2. Item (1) follows from $\gamma^U\;\in\;[\gamma,\Gamma]$. Items (2) and

(3) follows from

$$0 \;\leq\; -(E_t^Q[D_T])^{-1}\,\mathrm{Cov}_t^Q(\gamma^U(D_T)\,,\,D_T) \\ =\; -\,\frac{E_t^Q[\gamma^U(D_T)D_T]}{E_t^Q[D_T]} \;+\; E_t^Q[\gamma^U(D_T)] \;\leq\; -\gamma \,+\, \Gamma\,. \quad (41)$$

Similarly,

$$0 \;\leq\; -(E_t^Q[D_T])^{-1}\,\mathrm{Cov}_t^Q(\gamma^U(D_T)\,,\,W_{k\,T}) \\ =\; -\,\frac{E_t^Q[\gamma^U(D_T)W_{k\,T}]}{E_t^Q[W_{k\,T}]} \;+\; E_t^Q[\gamma^U(D_T)] \;\leq\; -\gamma \,+\, \Gamma \quad (42)$$

and items (2)-(3) immediately yield the required inequality for $\gamma_k > 1$. The case $\gamma_k < 1$ follows from the following easily verifiable identity

$$\pi_{k\,t} \;=\; \frac{\sigma\, e^{-b\,(T-t)}}{\sigma_t^S}\left(\gamma_k^{-1}\,\frac{E_t^Q[\gamma^U(D_T)\,W_{k\,T}]}{E_t^Q[W_{k\,T}]} \;-\; \mathrm{Cov}_t^Q(\gamma^U(D_T)\,,\,W_{k\,T})\right). \quad (43)$$

Q.E.D.

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