On the modeling of Debt Maturity and Endogenous Default: A caveat[*]

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Abstract We focus on structural models in corporate finance with roll-over debt structures in the vein of Leland (1994) and Leland and Toft (1996). We show that these models incorrectly assume that the optimal default is defined by the first time such that the firm's assets reaches a sufficiently low positive threshold that must be optimally determined. We characterize the optimal default policy and explain that the existing literature overestimates the probability of default and underestimates the equity value.

Keywords optimal stopping time. hitting time. endogenous default.

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1 Introduction.

Structural models in corporate finance allow an integrated analysis of firm, debt and equity values, optimal capital structure and decision to default. Particular attention has been devoted to the case where bankruptcy is endogenously triggered by limited liability equity-holders who set the default time so as to maximize the value of their claim. The equity-holders' problem consists then to maximize the option value associated to the irreversible decision to go bankrupt. The pioneering paper is Leland [16] which solves the problem when the debt is modeled as a consol bond, meaning a commitment to pay coupons indefinitely at some constant total coupon rate. In Leland [16], posing an infinite maturity for the debt guarantees a simple time-homogenous setting in which the equity-holders' problem takes the form of a standard perpetual American put option whose underlying asset is the value of the firm's assets, and the exercise price the present value of all coupon payments, net of tax shields. Computations are explicit and the optimal capital structure coming from the tradeoff between bankruptcy costs and tax shields can be fully studied in an analytical setting. Structural models with infinite debt maturity have been widely studied. Some important examples are Mella-Barral and Perraudin [21], Fan and Sundaresan [9], Goldstein, Ju and Leland [10] or Duffie and Lando [5] that extend Leland [16] in various thoughtful directions while maintaining the perpetual debt assumption.

To focus only on infinite life debt is clearly restrictive. Considering debt with finite maturity is not however a simple task as it breaks the stationary debt structure and precludes closed form solutions. In two influential papers, Leland [17] and Leland and Toft [19] circumvent this difficulty and propose a special debt structure aiming at studying debt with arbitrary finite maturity and endogenous default in a time homogenous environment that allows closed form solutions for the pricing of debt, firm and equity. The key point of their methodology is to consider a roll-over debt structure with regular repayments and renewals of principal and of coupon which should therefore guarantee a stationary debt structure.

Financial economists and applied mathematicians have derived numerous applications and extensions of this idea. Among leading papers, Leland [18] extends Leland [16] and studies the role of debt maturity on the incentives of equityholders' to increase the volatility of the firm's assets. Mauer and Ott [20] study in a setting à la Leland [16] the equity-holder's decision to invest in a growth opportunity. Eom, Helwege and Huang [7] confront Leland and Toft [19] to the data and find that, contrary to other structural models, the Leland and Leland and Toft models substantially overestimate the probability of default. Hackbarth, Miao and Morellec [11] uses Leland [16] to study the impact of macroeconomic conditions on credit risk and dynamic capital structure choice. Ericsson and Renault [8] rely on Leland and Toft [19] to develop a structural model with liquidity and credit risk. Hilberink and Rogers [12] and Kyprianou and Surya [14] extend the Leland and Leland and Toft methodology to Lévy processes. A common underlying assumption shared by this literature is that bankruptcy is triggered by equity-holders the first time such that the firm's asset reaches a sufficiently low positive constant threshold that must be optimally determined. In other words, the literature assumes that optimal bankruptcy policy is a hitting time.

The contribution of this paper is to show that, surprisingly, in the Leland and Leland and Toft methodology, the optimal bankruptcy policy is not a hitting time. The reason is

that the stream flow process for equity-holders implied by a roll-over debt structure is a function $h(t, V_t)$ of time and of the current value V_t of the firm's assets. It results that the optimal bankruptcy problem for the equity-holders is to solve a stopping problem of the type $\sup_\tau \mathbb{E}[\int_0^\tau e^{-rt} h(t, V_t) dt]$ which solution is proved not to be a hitting time. Precisely, we show that, in a roll over debt structure, the value of equity corresponds to the so called temporal section at date 0 of a bi-dimensional stopping problem in the (t, V) space. It follows that the optimal bankruptcy policy is defined by the first time such that the firm's assets process reaches a non constant time dependent boundary function. We derive our result for general roll-over debt structures (including those of Leland and Leland and Toft) and under general conditions for the process value of the firm's asset. The direct consequence of our finding is that the existing literature considers sub-optimal bankruptcy policies, overestimates the probability of default and underestimates the equity value. This of course shed some doubt on the relevance of results previously obtained on yields spreads, bond ratings, optimal capital structure and agency costs in a roll-over debt structure setting.

The outline of the paper is as follows. Section 1 describes the roll-over debt structure and the optimal default problem. Section 2 characterizes the optimal default policy which is proved to be not a trigger strategy. Section 3 applies our finding to the roll-over debt structure model of Leland [17]. Section 4 concludes.

2 Debt structure and optimal default.

For seek of clarity we first recall the original idea of Leland [17] and Leland and Toft [19]. To this purpose, we rely on Hilberink and Rogers [12] who proposed in section 2 of their paper a very clear and unified setting for the Leland and Leland and Toft methodology. At the end of this section we depart from the existing literature and we motivate our caveat on the modeling of debt maturity and endogenous default.

Following the way paved by structural models, we consider that equity and debt can be viewed as contingent claim's on the asset of the firm whose price dynamics is specified under a risk neutral probability measure $\mathbb{Q}$. Our assumptions on the process value of the firm's asset are those of the recent literature that extend Leland [17] to more general Lévy processes than the standard geometric brownian motion. Precisely, we start with a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q}; (\mathcal{F}_t)_{t \geq 0})$ where the filtration $(\mathcal{F}_t)_{t \geq 0}$ models the flow of information as the time goes by and satisfies the usual conditions: right-continuity and completion by $\mathbb{Q}$-negligible sets. In such a probability space, we assume that the dynamics of the firm's asset $(V_t)_{t \geq 0}$ is defined by equation

$$V_t = V e^{X_t}, \tag{1}$$

where $(X_t)_{t \geq 0}$ is a Lévy process with $X_0 = 0$. As usual, the firm generates cash flow at the rate δV_t at time t for some constant $\delta \in (0, \infty)$. It is assumed that the measure $\mathbb{Q}$ is a risk neutral probability measure meaning that the discounted value $e^{-(r-\delta)t} V_t$ of the firm's asset is a martingale, that is to say that

$$\mathbb{E}\left[e^{-(r-\delta)(u-t)} V_u | \mathcal{F}_t\right] = V_t,$$

for all $0 \leq t \leq u$, where r denotes the riskless rate. Our theoretical point is fully general and we do not need here to specify further the dynamics of the firm's asset to derive our main result[1].

We now turn to the description of the roll-over debt structure. The firm is partly financed by debt, which is being constantly retired and reissued in the following way. In time interval (t, t+ dt), the firm issues new debt with face value $f\,dt$ that will be paid back according to a maturity profile μ, where μ is a probability measure whose tail function will be denoted by

$$\Phi(x) = \int_x^{+\infty} \mu(dy).$$

Precisely, according to this debt structure, the amount of money the firm has to pay back in time interval $(t + s, t + s + ds)$ is $-f\,dt\,d\Phi(s)$. Now, bearing in mind all previously issued debt, at time 0 the amount of money the firm has to pay back in $(s, s + ds)$ is therefore

$$
\begin{aligned}
\left(\int_{-\infty}^0 -f\,dv\,d\Phi(s-v)\right) &= -d\left(\int_{-\infty}^0 f\,dv\,\Phi(s-v)\right) \\
&= -d\left(\int_s^\infty f\Phi(u)\,du\right) \\
&= f\Phi(s)\,ds.
\end{aligned}
\tag{2}
$$

Taking $s = 0$ in (2), we see that the face value of debt maturing in $(0, ds)$ is $f\,ds$, the same as the face value of the newly-issued debt. Thus the face value F of all pending debt is constant, equal to

$$F = f \int_0^\infty \Phi(s)\,ds.$$

Before maturity, bondholders receive coupons at rate c until default. Default happens at a stopping time τ whose determination is the core of this paper. Upon default, a fraction α of the value of the firm's assets is lost in reorganization. A bond issued at time 0 with coupon rate c, face value 1, maturity t and current value of the firm's asset V is worth

$$d_0(V, \tau, t) = \mathbb{E}\left[\int_0^{t \wedge \tau} c\,e^{-rs}\,ds\right] + \mathbb{E}\left[e^{-rt}\mathbb{1}_{t<\tau}\right] + \frac{1}{F}(1-\alpha)\mathbb{E}\left[V_\tau e^{-r\tau}\mathbb{1}_{\tau \leq t}\right]. \tag{3}$$

The first term on the right of (3) is interpreted as the net present value of all coupons paid up to date t or default time τ, whichever is sooner. The second term is the net present value of the principal repayment, if it occurs before bankruptcy. The final term is the net present value of what is recovered upon bankruptcy, if this happens before maturity. Indeed, V_τ is the value of the firm's assets when bankruptcy occurs and $(1 - \alpha)V_\tau$ is the value that remains after bankruptcy costs are deducted. Of this, the bondholder with face value 1 gets the fraction $\frac{1}{F}$, since its debt represents this fraction of the total outstanding.

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[1]To fix ideas, the reader may have in mind that $(X_t)_{t \geq 0}$ is a Lévy jump process as for instance in Chen Kou [2] or Dao Jeanblanc [4].

The total value at date 0 of all debt outstanding is

$$
\begin{aligned}
D(V,\tau) &= \int_0^\infty f\Phi(t)d_0(V,\tau,t)\,dt \\
&= fc\,\mathbb{E}\left[\int_0^\tau e^{-rs}\int_s^\infty \Phi(u)du\,ds\right] + f\,\mathbb{E}\left[\int_0^\tau e^{-rs}\Phi(s)ds\right] \\
&\quad + \frac{(1-\alpha)f}{F}\,\mathbb{E}\left[V_\tau\,e^{-r\tau}\int_\tau^\infty \Phi(u)\,du\right]
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
D(V,\tau) &= \mathbb{E}\left[\int_0^\tau e^{-rs}f\left(c\overline{\Phi}(s)+\Phi(s)\right)\,ds + \frac{(1-\alpha)f}{F}V_\tau e^{-r\tau}\overline{\Phi}(\tau)\right] \\
&= \mathbb{E}\left[\int_0^\tau e^{-rs}\frac{1}{\overline{\Phi}(0)}\left(C\overline{\Phi}(s)+F\Phi(s)\right)\,ds + (1-\alpha)V_\tau e^{-r\tau}\frac{\overline{\Phi}(\tau)}{\overline{\Phi}(0)}\right] \qquad (4)
\end{aligned}
$$

where $\overline{\Phi}(s) \equiv \int_s^\infty \Phi(u)\,du$ and $C \equiv cF$ is the total coupon rate.

Finally, note also that the expected maturity of each newly issued debt satisfies the relation

$$
\int_0^\infty t\mu(dt) = \int_0^\infty \Phi(u)du = \overline{\Phi}(0)
$$

We shall assume thereafter that the maturity profile μ is such that $\overline{\Phi}(0) < \infty$. In words, the debt profile has a finite expected maturity.

The above presentation of the debt structure, borrowed from Hilberink and Rogers [12], is very attractive since it encompasses both Leland and Toft [19] and the so called exponential model of Leland [16]. In Leland and Toft [19], we have $\mu(ds) = \delta_T(ds)$ where δ_T is the Dirac measure at T. This means that each new debt is issued with maturity $\overline{\Phi}(0) = \int_0^\infty s\mu(ds) = \int_0^\infty s\delta_T(ds) = T$. In the exponential model of Leland [16], the maturity profile is $\mu(ds) = me^{-ms}\,ds$ for some positive m. This means that the maturity of each new debt is chosen randomly according to an exponentially distributed random variable with mean $\overline{\Phi}(0) = \frac{1}{m}$. Because it requires easier computations, the exponential model has been widely used in the literature notably in Leland [18], Hilberink and Rogers [12], Chen and Kou [2], Dao and Jeanblanc [4], Hackbarth, Miao and Morellec [11], Kyprianou and Surya [14].

Always following this well established literature, we assume that there is a corporate tax rate θ, and that the coupon paid can be offset against tax. It follows an income stream of $(\delta V_t + \theta C)dt$ for the firm up to bankruptcy. At bankruptcy time τ, the firm's value amount to $(1-\alpha)V_\tau$ that is the value of the firm's assets minus bankruptcy costs. Formally, the value of firm for a given default time τ and a current value firm's asset V is

$$
v(V,\tau) = \mathbb{E}\left[\int_0^\tau e^{-rs}\left(\delta V_s + \theta C\right)\,ds + (1-\alpha)e^{-r\tau}V_\tau\right]; \qquad (5)
$$

4

or equivalently,

$$v(V, \tau) = V + \frac{\theta C}{r} \mathbb{E}[1 - e^{-r\tau}] - \alpha \mathbb{E}[e^{-r\tau} V_\tau]. \tag{6}$$

In terms of (4) and (5), the value of equity of the firm for bankruptcy policy τ is

$$
\begin{aligned}
E(V, \tau) &= v(V, \tau) - D(V, \tau) \tag{7} \\
&= \mathbb{E}\left[\int_0^\tau e^{-rs} \left(\delta V_s + C(\theta - \frac{\overline{\Phi}(s)}{\overline{\Phi}(0)}) - F\frac{\Phi(s)}{\Phi(0)} \right) ds + (1 - \alpha) e^{-r\tau} V_\tau (1 - \frac{\overline{\Phi}(\tau)}{\overline{\Phi}(0)}) \right] \\
&= \mathbb{E}\left[\int_0^\tau e^{-rs} h(s, V_s) \, ds \right],
\end{aligned}
$$

where the function $h(s, V_s)$ follows from a direct application of the Ito's lemma and is defined by the relation

$$h(s, V) \equiv V \left(\alpha\delta + (1 - \alpha)\frac{1}{\overline{\Phi}(0)}(\delta\overline{\Phi}(s) + \Phi(s)) \right) + C\theta - \frac{1}{\Phi(0)} \left(C\overline{\Phi}(s) + F\Phi(s) \right). \tag{8}$$

It is worth pointing out that the stream flow $h(s, V)$ paid to equity-holders depends both on time s and on the value of the firm's asset V_s.

Once the debt has been issued, the equity-holders' only decision is to select the default policy that maximizes the value of their claim. Equity-holders therefore chose the liquidation policy τ by solving the stopping problem

$$E(V) = \sup_{\tau \in \mathcal{T}_{0,\infty}} (v(V, \tau) - D(V, \tau)), \tag{9}$$

where $\mathcal{T}_{0,\infty}$ is the set of $(\mathcal{F}_t)$-stopping time with values in $[0, \infty]$. Following Leland [17] and Leland and Toft [19], there is a consensus for assuming that the optimal bankruptcy is defined by the first time such that the firm's assets V_t reaches a sufficiently low positive threshold that must be optimally determined. This is the reason why all the existing papers restrict the set of bankruptcy policies to hitting times $\tau_B = \inf\{t : V_t \leq B\}$ that are compatible with the limited liability condition[2]. This assumption leads the literature to set $\tau = \tau_B$ in equations (4), (5) and (7) and to reason in fact as if stopping problem (9) was equivalent to the optimization problem

$$\begin{cases} \max_{B \geq 0} (v(V, \tau_B) - D(V, \tau_B)), \\ v(V, \tau_B) - D(V, \tau_B) \geq 0, \quad \forall \ V \geq B \geq 0. \end{cases} \tag{10}$$

The contribution of this paper is to show that, surprisingly, it does not exist any maturity profile such that the optimal bankruptcy policy solution to (9) is a hitting time. Problems (9) and (10) are thus not equivalent and the existing literature, by focusing on (10), considers sub-optimal bankruptcy policies and therefore underestimates the value of equity.

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[2]The value of equity remains non-negative at all times. Note that, taking $\tau = 0$ in (7) yields to $E(V, \tau)_{\tau=0} = 0$ from which it follows that the value of equity (9) remains non-negative for all value of the firm's assets as it must be.

3 The main result

Our main result characterizes the optimal bankruptcy policy solution to (9). We show the following.

Theorem 1 *There exists a non constant boundary function $b^*(.)$ defined on $[0, \infty)$ such that $\tau^* = \inf\{t : V_t \leq b^*(t)\}$ is an optimal stopping time for problem (9).*

The immediate consequence of Theorem 1 is that, since the boundary function b^* is not constant, the optimal stopping time τ^* solution to (9) is not a hitting time. Our result is obtained for any maturity profile μ and for any firm's asset process $(V_t)_{t \geq 0}$ defined in section 2.

Before starting with the proof of our Theorem, it is useful to re-formulate the value of equity for a given bankruptcy policy τ. We deduce from equations (4) and (6) that

$$
\begin{aligned}
E(V, \tau) &= v(V, \tau) - D(V, \tau) \\
&= V - \int_0^\infty e^{-rs} \left(C \frac{\overline{\Phi}(s)}{\overline{\Phi}(0)} + F \frac{\Phi(s)}{\overline{\Phi}(0)} - \theta C \right) ds \\
&\quad + \mathbb{E}\left[e^{-r\tau} (\beta(\tau) - \gamma(\tau) V_\tau) \right],
\end{aligned}
$$

with

$$
\beta(t) = \int_0^\infty e^{-rs} \left(C \frac{\overline{\Phi}(t+s)}{\overline{\Phi}(0)} + F \frac{\Phi(t+s)}{\overline{\Phi}(0)} - \theta C \right) ds,
$$

and

$$
\gamma(t) = \alpha + (1 - \alpha) \frac{\overline{\Phi}(t)}{\overline{\Phi}(0)}.
$$

It then results the following re-statement of problem (9)

$$
\begin{aligned}
E(V) &= V - \int_0^\infty e^{-rs} \left(C \frac{\overline{\Phi}(s)}{\overline{\Phi}(0)} + F \frac{\Phi(s)}{\overline{\Phi}(0)} - \theta C \right) ds \\
&\quad + \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-r\tau} (\beta(\tau) - \gamma(\tau) V_\tau) \right].
\end{aligned}
\tag{11}
$$

that is, the value of equity $E(V)$ is equal equal to the equity value if bankruptcy is never declared, $\left(V - \int_0^\infty e^{-rs} (C \frac{\overline{\Phi}(s)}{\overline{\Phi}(0)} + F \frac{\Phi(s)}{\overline{\Phi}(0)} - \theta C) \, ds \right)$, *plus* the value of the option to declare bankruptcy at any time which is defined by stopping problem

$$
P(V) \equiv \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-r\tau} g(\tau, V_\tau) \right],
\tag{12}
$$

where

$$
g(t, V) \equiv \beta(t) - \gamma(t) V.
$$

Saying it differently, as emphasized in the introduction, the equity holder's problem consists to maximize the option value associated to the irreversible decision to go bankrupt. We

now turn to the proof of Theorem 1 which goes through three main steps and relies on the analysis of stopping problem (12). Step 1 establishes that problem (12) corresponds to the so called 0-section of a bi-dimensional stopping problem in the (t, V) space. Step 2 proves that the associated stopping region is defined by a boundary function $b^*(t)$. Step 3 shows that this boundary function is not constant.

Step 1. *let $\mathcal{P} : [0, \infty) \times (0, \infty) \longrightarrow \mathbb{R}$ be the value function of the Markov bi-dimensional optimal stopping problem in the (t, V) space*

$$\mathcal{P}(t, V) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-r\tau} g(t + \tau, V_\tau^V) ds \right] \tag{13}$$

then, the option value to declare bankruptcy satisfies

$$P(V) = \mathcal{P}(0, V). \tag{14}$$

Furthermore the stopping time

$$\tau^* \equiv \inf\{s \geq 0 \,|\, \mathcal{P}(t + s, V_s) = g(t + s, V_s)\} \tag{15}$$

is optimal for problem (13).

<u>Proof of Step 1</u> We follow the optimal stopping theory[3] and consider the Snell envelope process

$$(\mathcal{P}_t)_{t \geq 0} \equiv \left(\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,\infty}} \mathbb{E}\left[e^{-r\tau} g(\tau, V_\tau) \, ds \,|\, \mathcal{F}_t \right] \right)_{t \geq 0} \tag{16}$$

which is the smallest supermartingale that dominates $\left(e^{-rt} g(t, V_t) \right)_{t \geq 0}$. We have that

$$\begin{aligned}
\mathcal{P}_t &= e^{-rt} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}[e^{-r\tau} g(t + \tau, V_{t+\tau}) | \mathcal{F}_t] \\
&= e^{-rt} \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}[e^{-r\tau} g(t + \tau, V_\tau^{V_t})] \\
&= e^{-rt} \mathcal{P}(t, V_t),
\end{aligned}$$

where the second equality comes from the strong Markov property. It therefore results that, as announced

$$P(V) \equiv \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}[e^{-r\tau} g(\tau, V_\tau^V)] = \mathcal{P}(0, V). \tag{17}$$

Moreover, according to the optimal stopping theory again, the process $\left(\mathcal{P}_{u \wedge \tau_t^*} \right)_{u \geq t}$ is a $\mathcal{F}_u$-martingale with

$$\tau_t^* \equiv \inf\{u \geq t \,|\, \mathcal{P}_u = e^{-ru} g(u, V_u)\}.$$

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[3]We refer to El Karoui [6] and Karatzas and Shreve [13], Appendix D Theorem D9 and Theorem D12. See also Theorems 10.1.9 and 10.1.12 by Oksendal [22], for a detailed presentation of optimal stopping theory.

Therefore, we have that for every integer n,

$$
\begin{aligned}
\mathcal{P}_t &= \mathbb{E}\left[\mathcal{P}_{\tau_t^*}\mathbb{1}_{\{\tau_t^* \leq n\}}|\mathcal{F}_t\right] + \mathbb{E}\left[\mathcal{P}_n\mathbb{1}_{\{\tau_t^* \geq n\}}|\mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[e^{-r\tau_t^*}g(\tau_t^*, V_{\tau_t^*})\mathbb{1}_{\{\tau_t^* \leq n\}}|\mathcal{F}_t\right] + e^{-rn}\int_0^\infty e^{-ru}\left(C\frac{\overline{\Phi}(u)}{\overline{\Phi}(0)} + F\frac{\Phi(u)}{\overline{\Phi}(0)}\right)du, \quad (18)
\end{aligned}
$$

where the last inequality results from the relation

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{P}_n\mathbb{1}_{\{\tau_t^* \geq n\}}|\mathcal{F}_t\right] &\leq e^{-rn}\mathbb{E}\left[\mathcal{P}(n, V_n)|\mathcal{F}_t\right] \\
&\leq e^{-rn}\beta(n) \leq e^{-rn}\int_0^\infty e^{-ru}\left(C\frac{\overline{\Phi}(u)}{\overline{\Phi}(0)} + F\frac{\Phi(u)}{\overline{\Phi}(0)}\right)du.
\end{aligned}
$$

Because $\overline{\Phi}(u) < \overline{\Phi}(0) < \infty$, letting n tend to infinity in (18), we obtain

$$
\mathcal{P}_t \leq \mathbb{E}\left[e^{-r\tau_t^*}g(\tau_t^*, V_{\tau_t^*})|\mathcal{F}_t\right].
$$

Because the reverse inequality always holds, we get

$$
\mathcal{P}_t = \mathbb{E}\left[e^{-r\tau_t^*}g(\tau_t^*, V_{\tau_t^*}^V)|\mathcal{F}_t\right]. \tag{19}
$$

Using the Strong Markov property and the equality $\mathcal{P}_t = e^{-rt}\mathcal{P}(t, V_t)$, we get

$$
\begin{aligned}
\mathcal{P}(t, V_t) &= \mathbb{E}\left[e^{-r(\tau_t^*-t)}g(\tau_t^*, V_{\tau_t^*})|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[e^{-r\tau^*}g(t+\tau^*, V_{\tau^*}^{V_t})\right]
\end{aligned}
$$

with τ^* is defined by Equation (15).

Step 2. *Let us define the stopping region of problem (13) by*

$$
S \equiv \{(t, V) \in [0, \infty[^2 \,|\, \mathcal{P}(t, V) = g(t, V)\}
$$

and its t-sections by

$$
S_t \equiv \{V \in [0, \infty[\,|\, \mathcal{P}(t, V) = g(t, V)\}.
$$

Note that the stopping region S can be written $S = \cup_{t\geq0}\{t\} \times S_t$ and that the optimal stopping time τ^* is the first time when the process (t, V_t) hits the stopping region S. Step 2 shows that the t-sections are left-connected, in other words there is a function b^* defined on $(0, \infty)$ such that for every $t \geq 0$, $S_t =]0, b^*(t)]$.

<u>Proof of Step 2</u> We have to prove that, if $\mathcal{P}(t, \tilde{V}) = g(t, \tilde{V})$ then $\mathcal{P}(t, V) = g(t, V)$ for all $V \leq \tilde{V}$. Taking advantage from the relation $V_t^V = V_t^{\tilde{V}} - V_t^{\tilde{V}-V}$ deduced from (1), we obtain that

$$
\mathbb{E}\left[e^{-r\tau}(\beta(t+\tau) - \gamma(t+\tau)V_\tau^V)\right] \leq \mathbb{E}\left[e^{-r\tau}(\beta(t+\tau) - \gamma(t+\tau)V_\tau^{\tilde{V}})\right] + \mathbb{E}\left[e^{-r\tau}\gamma(t+\tau)V_\tau^{\tilde{V}-V}\right]. \tag{20}
$$

An easy calculus based on the martingale property of the process $e^{-(r-\delta)}V_t$ and on the fact that the positive function $\gamma(.)$ is decreasing yields to

$$
E\left[e^{-r\tau}\gamma(t+\tau)V_\tau^{\tilde{V}-V}\right] \leq E\left[e^{-\delta\tau}\gamma(t)(\tilde{V} - V)\right] \leq \gamma(t)(\tilde{V} - V). \tag{21}
$$

8

Using (21) and taking the supremum over τ in (20) yields to

$$\mathcal{P}(t,V) \leq \mathcal{P}(t,\tilde{V}) + \gamma(t)(\tilde{V} - V).$$

Since $\tilde{V} \in S_t$, we obtain $\mathcal{P}(t,\tilde{V}) = \beta(t) - \gamma(t)\tilde{V}$ and thus,

$$\mathcal{P}(t,V) \leq \beta(t) - \gamma(t)V,$$

from which we deduce that $V \in S_t$ since by definition $\mathcal{P}(t,V) \geq \beta(t) - \gamma(t)V$. It thus follows that S_t is an interval $(0, b^*(t)]$ where $b^*(t)$ is defined as $\sup\{V \in (0,\infty) : (t,V) \in S\}$. Moreover, the optimal stopping time τ^* can be expressed as

$$\tau^* = \inf\{s \geq 0 \,|\, V_s \leq b^*(t+s)\}. \tag{22}$$

Remark that, taking $t = 0$ in (22) gives the optimal stopping time solution to (17) and therefore to our main problem (9). To prove Theorem 1, it remains to show in a last step that the function b^* is not constant.

Step 3. *The optimal boundary function b^* is not constant.*

Proof of Step 3

Because $\Phi(t)$ and $\overline{\Phi}(t)$ are decreasing and tend to 0 when t goes to infinity, we have that $\beta(t) \leq 0$, for all $t \geq \bar{t}$ where $\bar{t}$ is implicitly defined by

$$C\frac{\overline{\Phi}(\bar{t})}{\overline{\Phi}(0)} + F\frac{\Phi(\bar{t})}{\overline{\Phi}(0)} - C\theta = 0. \tag{23}$$

Thus, for all $t \leq \bar{t}$, $g(t,V) = \beta(t) - \gamma(t)V \leq 0$ for all $V > 0$. Therefore, for all $t \geq \bar{t}$, $\mathcal{P}(t,V) \leq 0$ as the supremum of non positive real numbers. Now, for $s \geq 0$, we have that

$$\mathcal{P}(t,V) \geq e^{-rs}\beta(s+t) - \gamma(s+t)\mathbb{E}[e^{-rs}V_s^V].$$

Letting s tend to infinity yields that $\mathcal{P}(t,V) \geq 0$. Therefore, $\mathcal{P}(t,V) = 0$ for all $t \geq \bar{t}$ which implies that $S_t = \emptyset$ for all $t \geq \bar{t}$, or equivalently that $b^*(t) = 0$ for all $t \geq \bar{t}$.

We now show that $S \neq \emptyset$, from which it will result that there exists $t \in [0,\bar{t})$ such that $b^*(t) > 0$ and thus will end the proof of step 3. We proceed by way of contradiction. Assume $S = \emptyset$ then equityholders would optimally postpone bankruptcy and $\tau = \infty$ would be optimal for problem (17). In that case, the value of firm (5) would be

$$v(V,\infty) = \int_0^\infty e^{-rs}(\delta V_s + \theta C)\,ds] = V + \frac{\theta C}{r},$$

and the total value of debt (4) would simplify to

$$
\begin{aligned}
D(V,\infty) &= \int_0^\infty e^{-rs}\frac{1}{\overline{\Phi}(0)}\left(C\overline{\Phi}(s) + F\Phi(s)\right)ds \\
&= \frac{C}{r} + (F - \frac{C}{r})\frac{1}{\overline{\Phi}(0)}\int_0^\infty e^{-rs}\Phi(s)\,ds.
\end{aligned}
\tag{24}
$$

Taking into account the standard constraint[4] that the total coupon C is set so the market value of debt equals principal value F, we obtain from (24) the relation $F = \frac{C}{r}$. It then follows the market value of the equity of the firm for a policy that consists in never going bankrupt:

$$
\begin{aligned}
E(V, \infty) &= v(V, \infty) - D(V, \infty) \\
&= V - (1 - \theta)\frac{C}{r}.
\end{aligned}
$$

Clearly, $E(V, \infty) = V - (1 - \theta)\frac{C}{r}$ is not positive for all $V > 0$ and cannot be therefore the value function of equity. It follows that $S \neq \emptyset$ from which it results that $b^*(t) > 0$ for some $t \in [0, \bar{t})$. This concludes the proof of Theorem 1.

Summing up our findings, it therefore follows from Theorem 1 and equation (11) that the value of equity solution to problem (9) satisfies

$$
E(V) = \begin{cases}
0 & \forall\, V \leq b^*(0), \\
V - \displaystyle\int_0^\infty e^{-rs}\left(C\frac{\overline{\Phi}(s)}{\overline{\Phi}(0)} + F\frac{\Phi(s)}{\overline{\Phi}(0)} - \theta C \right) ds + P(V), & \forall\, V \geq b^*(0),
\end{cases} \tag{25}
$$

where
$$
P(V) = \mathbb{E}\left[e^{-r\tau^*}(\beta(\tau^*) - \gamma(\tau^*)V_{\tau^*}) \right] \quad \text{with} \ \ \tau^* = \inf\{s : V_s \leq b^*(s)\}. \tag{26}
$$

We have therefore shown that, in the roll-over structure of debt initiated by Leland and Leland and Toft the equity-holders problem corresponds to the 0-section of a bi-dimensional stopping problem in the (t, V) space. Equation (25) enlightens the crucial point missed by the literature: the value of the firm's asset is a sufficient statistics only for the decision problem whether or not to go bankrupt but the pricing of equity involves a stopping time build on a time-dependent boundary function. That is, at each instant of time equity-holders optimally go bankrupt if the current value of the firm's assets is below the constant threshold $b^*(0)$. However, the optimal bankruptcy policy τ^* that solves the equity-holders' problem is not defined by the hitting time $\inf\{t : V_t \leq b^*(0)\}$ but by the first time at which the process $(V_t)_{t \geq 0}$ reaches the non constant boundary function $b^*(.)$ as emphasized by (25) and (26).

It is also worth pointing out that, by virtue of stopping theory, we have the inequality $b^*(0) \leq B^*$, where the threshold B^* is the solution to problem (10) derived in the standard literature. This inequality says that the existing papers, by considering sub-optimal trigger strategies, have overestimated the probability of default[5]. The optimality criterion for B^* solution to (10) found in the corporate finance literature is equation

$$
\frac{\partial E}{\partial V}(V, \tau_{B^*})_{|V=B^*} = 0, \tag{27}
$$

which is usually referred as a "smooth pasting condition". This expression is actually very misleading. Indeed, the so called smooth pasting condition relies on optimal stopping theory

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[4]For contractual purposes, it is assumed in the literature that debt is issued at par. This classical assumption is convenient but clearly not crucial for our result.

[5]Note that this is in line with the empirical findings by Eom, Helwege and Huang [7].

and reflects the C^1 nature of the value function of a stopping problem across the optimal boundary. The vocabulary "smooth pasting condition" is thus here not appropriate since, as shown in Theorem 1, problem (10) restricts bankruptcy policy to sub-optimal hitting times. The point is that the smooth-pasting condition would characterize the optimal bankruptcy policy if the optimal bankruptcy policy was a hitting time[6]. This is what happens in studies that consider a perpetual coupon bond. Again, in the setting of a roll-over debt structure, the optimal bankruptcy time is not a hitting time and equation (27) is of no help for solving the equity-holders' problem (9).

4   Illustration: The exponential model

To illustrate our theoretical point, we consider thereafter the so-called exponential model of Leland [17]. The value of the firm's asset follows a standard geometric Brownian motion (that is $X_t = (r - \delta - \frac{\sigma^2}{2}) + \sigma W_t$ under $\mathbb{Q}$) and the maturity profile has an exponential law with parameter $m > 0$. We have therefore $\mu(dt) = me^{-mt}dt$ and $\Phi(t) = m\overline{\Phi}(t) = e^{-mt}$. The quantity $\overline{\Phi}(0) = \frac{1}{m}$ is interpreted as the average maturity of the debt. The limit case $m = 0$ corresponds to infinite debt maturity and when m tends to infinity, the average maturity of debt approaches zero (likely short term maturity debt). In this setting, the equity value (25) becomes

$$E(V) = \begin{cases} 0 & \forall\, V \leq b^*(0), \\ V + \frac{\theta C}{r} - \frac{C + mF}{r + m} + P(V), & \forall\, V \geq b^*(0), \end{cases} \tag{28}$$

with $P(V) = \mathcal{P}(0, V)$ where $\mathcal{P}$ is the value function of stopping problem (13) with $\beta(t) = \frac{C + mF}{r + m}e^{-mt} - \frac{\theta C}{r}$ and $\gamma(t) = \alpha + (1 - \alpha)e^{-mt}$.

On the other hand, restricting the set of bankruptcy policy to hitting time $\tau_B = \inf\{t : V_t \leq B\}$ and solving problem (10) yields to the classical formula for the equity value derived in the literature[7]

$$E(V, \tau_{B^*}) = \begin{cases} 0 & \text{if } V \leq B^*, \\ V + \frac{\theta C}{r} - \frac{C + mF}{r + m} - (\alpha B^* + \frac{\theta C}{r})\left(\frac{V}{B^*}\right)^{\xi_0} + (\frac{C + mF}{r + m} - (1 - \alpha)B^*)\left(\frac{V}{B^*}\right)^{\xi_m} & \text{if } V > B^*. \end{cases} \tag{29}$$

where ξ_m is the negative root of the quadratic equation $\frac{1}{2}\sigma^2 \xi(\xi - 1) + (r - \delta)\xi - (r + m) = 0$ and where

$$B^* \equiv \frac{\xi_0 \frac{\theta C}{r} - \xi_m \frac{C + mF}{r + m}}{1 - \xi_m + \alpha(\xi_m - \xi_0)} \tag{30}$$

solves equation (27).

A straightforward calculus shows that, when $m = 0$, equations (28) and (29) coincide and we are coming back to the case of an infinite debt maturity studied in Leland [16] in

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[6]For a recent contribution on threshold strategies and smooth pasting principle, we refer to Villeneuve [23] whose Theorem 4.2 and Proposition 4.6 give simple conditions under which an optimal stopping time is a hitting time. See also Brekke and Øksendal [1] who show that when the optimal stopping time associated to a stopping problem is a hitting time then, the smooth pasting condition is obtained as a direct consequence of a standard first order condition and characterizes the optimal threshold.

[7]See for instance Leland [17], equations (17) and (19) pages 15 and 16.

which the optimal bankruptcy policy is indeed a hitting time. When $m > 0$, the formula (29) underestimates the equity value whose correct formula is (28). We are able to derive a closed form formula for the difference between the value functions (28) and (29) for very short term debt, that is when m tends to infinity. We show the following Proposition proved in the appendix.

Proposition 1 *The limit when m goes to infinity of the value functions (28) and (29) are respectively*

$$E(V) = \begin{cases} 0 & \text{if } V \leq b^*(0), \\ V + \frac{\theta C}{r} - F & \text{if } V > b^*(0), \end{cases} \tag{31}$$

where $b^*(0) = F - \frac{\theta C}{r}$, and

$$E(V, \tau_{B^*}) = \begin{cases} 0 & \text{if } V \leq B^*, \\ V + \frac{\theta C}{r} - F - \left(\alpha \frac{F}{1-\alpha} + \frac{\theta C}{r}\right)\left(\frac{V(1-\alpha)}{F}\right)^{\xi_0} & \text{if } V > B^*, \end{cases} \tag{32}$$

where $B^* = \frac{F}{1-\alpha}$.

It is easy to interpret formulae (31) and (32). Because the maturity of each outstanding debt is very short, the option value to go bankrupt is worthless. It follows that the value of equity is simply equal to the current value of the firm's asset *plus* the discounted tax shields due to the coupon payments *minus* the market value of debt which is equal to its face value[8] F. This therefore leads to the bankruptcy threshold $b^*(0) = F - \frac{\theta C}{r}$. By restricting the set of bankruptcy policies to hitting times, the existing literature yields to formula (32) which exhibits a larger bankruptcy threshold $B^* = \frac{F}{1-\alpha} > b^*(0) = F - \frac{\theta C}{r}$ and a negative option value to go bankrupt equal to $-\left(\alpha \frac{F}{1-\alpha} + \frac{\theta C}{r}\right)\left(\frac{V(1-\alpha)}{F}\right)^{\xi_0}$. This latter expression corresponds to the underestimation of the equity value in the exponential model for very short maturities when the default policy is assumed to be a hitting time. The next paragraph quantifies the undervaluation of equity for various maturity parameters m. This requires a numerical implementation.

Numerical Implementation

The exponential model offers a simple setting in which well-known numerical methods related to partial differential equations can be implemented. It indeed follows from a standard reasoning[9] that $\mathcal{P}$ solution to problem (13) solves the variational inequalities

$$\max\left(\frac{\partial \mathcal{P}}{\partial t} + \mathcal{A}\mathcal{P}, g - \mathcal{P}\right) = 0 \tag{33}$$

---

[8]In the limit case $m = \infty$, the maturity of each outstanding debt is very short and the debt is always issued at par. Formally, the market value of debt (4) written in the exponential model tends to F when m tends to ∞.

[9]See for instance Theorem 10.3.10 of Lamberton [15] whose assumptions are satisfied in our setting. The crucial point is to remark that the partial derivatives $\frac{\partial \mathcal{P}}{\partial V}$ and $\frac{\partial \mathcal{P}}{\partial t}$ are locally bounded functions on $[0, \infty) \times (0, \infty)$. It can be indeed proved without any difficulties that $||\frac{\partial \mathcal{P}}{\partial V}||_{L^\infty[0,\infty)\times(0,\infty)} \leq 1$ and $||\frac{\partial \mathcal{P}}{\partial t}||_{L^\infty[0,\infty)\times(0,\infty)} \leq K_1 + K_2 V$ for some positive constant K_1 and K_2.

in the open set $[0, \bar{t}) \times (0, \infty)$ with the terminal condition

$$\mathcal{P}(\bar{t}, .) = 0, \tag{34}$$

where $\mathcal{AP} = \frac{\sigma^2}{2} \frac{\partial^2 \mathcal{P}}{\partial V^2} V^2 + (r - \delta) \frac{\partial \mathcal{P}}{\partial V} \partial V - r\mathcal{P}$ and where $\bar{t} = \frac{1}{m} \ln \frac{r(C+mF)}{\theta C(r+m)}$ is the solution to equation (23). The numerical procedure that we now use for computing the equity value (28) is standard and relies on the variational inequalities (33) and (34). We take the values of the parameters as follows[10]: $\sigma = 0.2$, $r = 7.5\%$, $\delta = 7\%$, $\alpha = 50\%$, $\theta = 35\%$, $V = 100$, $F = 40$ and $C = 5$. We then compare the value functions (28) and (29) for various average maturities: 3 months ($m = 4$), 6 months (m= 2), 1 year (m=1), 5 years (m= 0.2), 10 years (m =0.1), 20 years (m = 0.05). We run the limit case $m = \infty$ by considering $m = 1000000$. We also compute for each average maturity the thresholds $b^*(0)$ and B^*. Table 1 reports our results.

Table 1.

m	∞	4	2	1	0.2	0.1	0.05
$E(V, \tau_{B^*})$	39.05	59.20	63.30	66.83	69.36	67.57	65
$E(V)$	83.33	82.84	82.37	81.47	76.06	71.90	67.34
B^*	79.9	60.08	54.40	48.15	34.87	31.20	29
$b^*(0)$	16.67	17.13	17.56	18.38	22.65	25.1	26.3

Since (28) and (29) coincide when $m = 0$, it is not surprising to observe for large average maturities (m close to 0), small differences between $E(V)$ and $E(V, \tau_{B^*})$ (this difference amount to 3.47% of the equity value $E(V)$ for an average maturity of 20 years). The striking feature of Table 1 is the gap between the two value functions (28) and (29) for intermediate and short average maturities. For instance, for a three month average maturity, the difference between $E(V)$ and $E(V, \tau_{B^*})$ amounts to 28% of $E(V)$ the value of equity. The differences between the two bankruptcy thresholds are also dramatic for short maturities. The values obtained for m=1 and m= 2 are in line with the analysis of the limit case $m = \infty$. In particular the equity value $E(V)$ derived from the stopping problem (9) tends to $V + \frac{\theta C}{r} - F = 83.33$ when m tends to infinity, a limit which is almost reached for $m = 1$. The convergence of $E(V, \tau_{B^*})$ to its limit is slower (the limit of $E(V, \tau_{B^*})$ when m tends to infinity is computed from (32) and is equal to 38.32). Overall, our numerical results suggest that, the exponential model dramatically overestimates the probability of default and underestimates the value of equity for short maturities whereas the problem is less severe for long term debt.

5   Conclusion

Corporate models with roll-over debt structures looked very attractive for studying endogenous default with arbitrary finite debt maturity in a time homogenous framework. We show

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[10]Our objective here is simply to illustrate the difference between $E(V)$ and $E(V, \tau_{B^*})$ for a given set of parameters. We do not assume that debt is issued at par. Note however that, in the limit case $m = \infty$, the debt is by construction always issued at par as remarked in footnote 8.

that, surprisingly, roll-over debt structures do not reach this objective: We prove that the optimal default policy in such a setting is not defined by a simple trigger strategy but follows from a bi-dimensional stopping problem in the (t, V) space that we fully characterize. The consequence of our finding is that, unfortunately the existing literature has considered up to now sub-optimal default policies. To date, the analysis of corporate structural models dealing with endogenous default and finite maturity debt cannot be done in a time homogenous framework and requires numerical procedures.

6   Appendix

Proof of Proposition 1

Formula (32) is easily deduced from (29) and (30) by taking the limit when m tends to ∞. We now index the value function $\mathcal{P}$ by the maturity parameter m and we compute for all $t \geq 0$ and for all $V > 0$, the limit of $\mathcal{P}_m(t, V)$ when m tends to infinity. A straightforward calculus leads to the inequality

$$\beta(t + \tau) - \gamma(t + \tau)V_\tau^V \leq e^{-m\tau}(\beta(t) - \gamma(t)V_\tau^V),$$

$\mathbb{P}$-almost surely for all $\tau \in \mathcal{T}_{0,\infty}$. We then deduce that

$$0 \leq \mathcal{P}_m(t, V) \leq \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-(r+m)\tau}(\beta(t) - \gamma(t)V_\tau^V)\right] \equiv \overline{\mathcal{P}}_m(t, V) \tag{35}$$

The calculus of $\overline{\mathcal{P}}_m(t, V)$ is similar to the one of a perpetual American put option and yields to

$$\overline{\mathcal{P}}_m(t, V) = \begin{cases} \beta(t) - \gamma(t)V & \text{if } V \leq \bar{b}_m(t) = \frac{\beta(t)}{\gamma(t)}\frac{\xi_m}{\xi_m - 1}, \\ (\beta(t) - \gamma(t)\bar{b}_m(t))\mathbb{E}[e^{-(r+m)\tau_{\bar{b}_m(t)}}] & \text{if } V \geq \bar{b}_m(t) = \frac{\beta(t)}{\gamma(t)}\frac{\xi_m}{\xi_m - 1}. \end{cases}$$

where $\tau_{\bar{b}_m(t)} = \inf\{t : V_t \leq \bar{b}_m(t)\}$. Let us recall that $\beta(t) = \frac{C+mF}{r+m}e^{-mt} - \frac{\theta C}{r}$ which implies that, for m sufficiently large, $\bar{b}_m(t)$ is strictly negative. It follows that, for m sufficiently large, $\overline{\mathcal{P}}_m(t, V) = (\beta(t) - \gamma(t)\bar{b}_m(t))\mathbb{E}_V[e^{-(r+m)\tau_{\bar{b}_m(t)}}] = 0$. It then results from (35) that $\lim_{m \longrightarrow \infty} \mathcal{P}_m(t, V) = 0 \ \forall t \geq 0, \forall V > 0$ and thus $\lim_{m \longrightarrow \infty} \mathcal{P}_m(0, V) = 0, \ \forall V > 0$. Formula (31) is then obtained from equation (28).

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